كلية الحاسبات والذكاء الإصطناعي

BS111
Probability and Statistics

Lecture 11

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2021

Ch 3.2: Continuous Prob. Distr.

- Continuous Uniform Distribution.
- Normal Distribution.

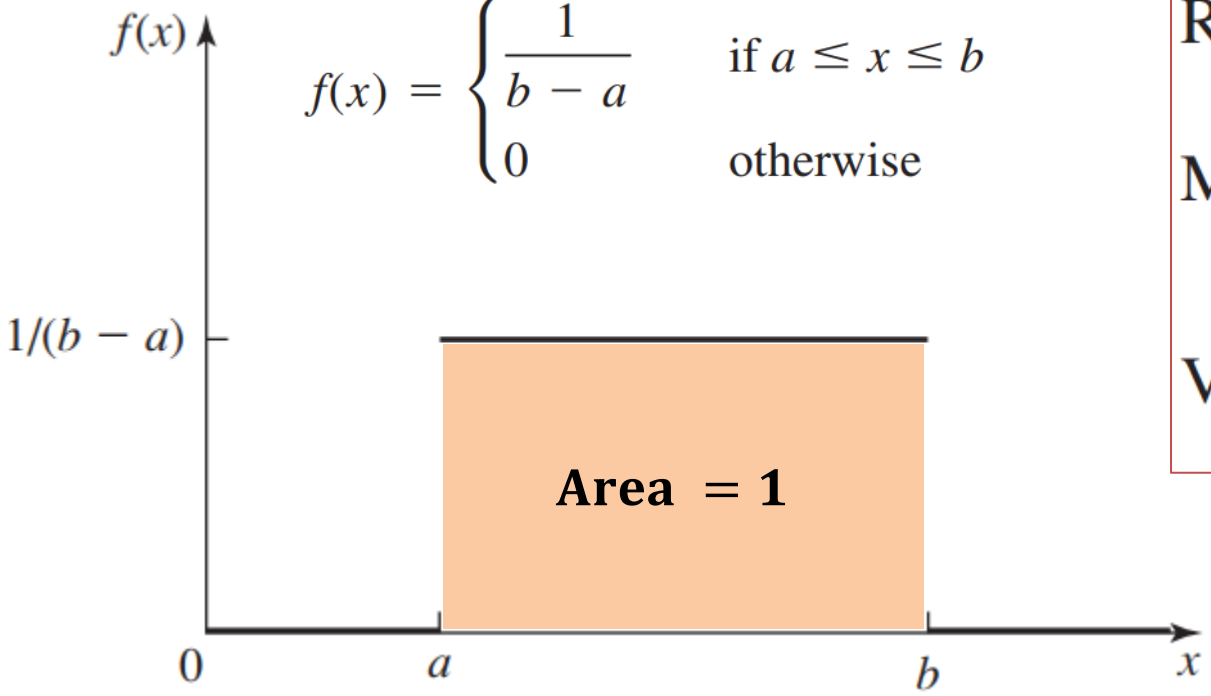
Continuous Uniform Dist. (1/4)

Probability Density Function:

$U(a, b)$

- a and b real numbers with $a < b$.

$$f(x) = \begin{cases} \dfrac{1}{b-a} & \text{if } a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$



Range	$[a, b]$
Mean	$\dfrac{a+b}{2}$
Variance	$\dfrac{(b-a)^2}{12}$

Continuous Uniform Dist. (2/4)

Cumulative Distribution Function:

$U(a, b)$

$$\text{Density} \qquad f(x) = \begin{cases} \dfrac{1}{b-a} & \text{if } a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

$$F(x) = \int_a^x \frac{1}{b-a} du = \frac{x-a}{b-a}$$

cumulative distribution function

$P(X \le x)$

Continuous Uniform Dist. (2/4)

Cumulative Distribution Function:

$U(a, b)$

$$\text{Density} \quad f(x) = \begin{cases} \dfrac{1}{b-a} & \text{if } a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

$$\text{Distribution} \quad F(x) = \begin{cases} 0 & \text{if } x < a \\ \dfrac{x-a}{b-a} & \text{if } a \le x \le b \\ 1 & \text{if } b < x \end{cases}$$

Continuous Uniform Dist. (3/4)

Example1 (1/6):

Suppose that a large conference room at a certain company can be reserved for no more than 4 hours. Both long and short conferences occur quite often. In fact, it can be assumed that the length X of a conference has a uniform distribution on the interval [0, 4].

(a) What is the probability density function?

(b) What is the probability that any given conference lasts at least 3 hours?

Continuous Uniform Dist. (3/4)

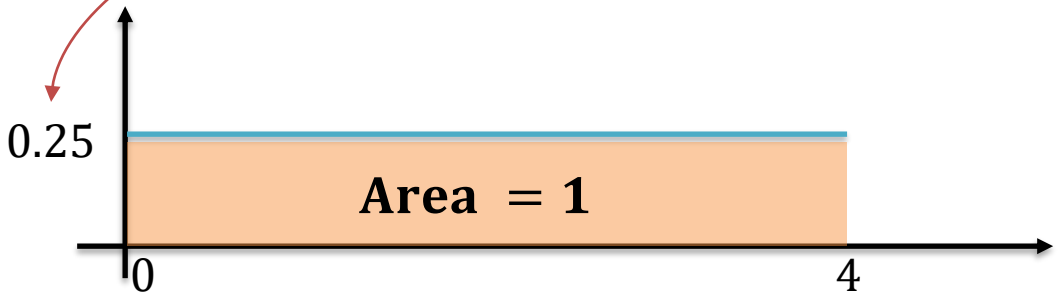
Example1 (2/6):

$X \sim U(0, 4)$

$$f(x) = \begin{cases} \dfrac{1}{b-a} & \text{if } a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

a) What is the probability density function?

$$f(x) = \begin{cases} \frac{1}{4}, & 0 \le x \le 4, \\ 0, & \text{elsewhere.} \end{cases}$$



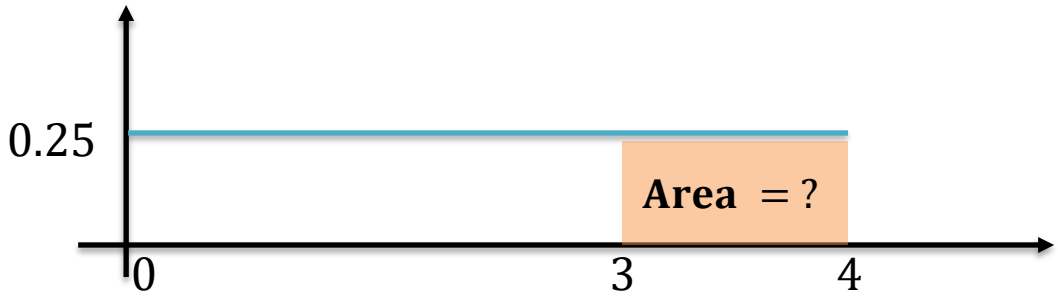
Continuous Uniform Dist. (3/4)

Example1 (3/6):

$\boldsymbol{X \sim U(0,4)}$

b) What is the probability that any given conference lasts at least 3 hours?

$$P(X \geq 3) = \int_3^4 \frac{1}{4}\, dx$$



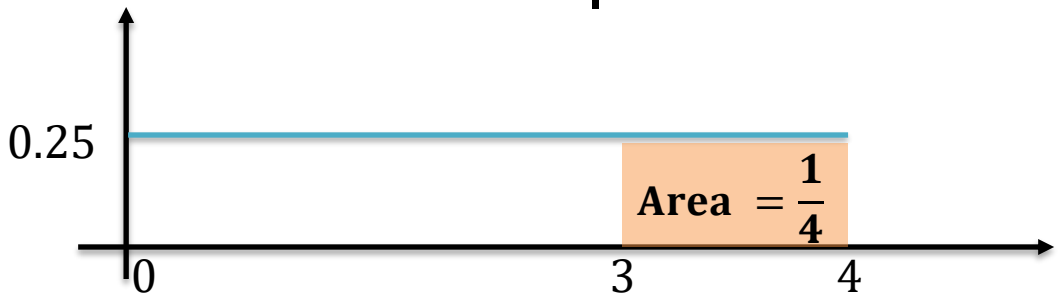
Continuous Uniform Dist. (3/4)

Example1 (4/6):

$\boldsymbol{X \sim U(0,4)}$

b) What is the probability that any given conference lasts at least 3 hours?

$$P(X \geq 3) = \int_3^4 \frac{1}{4}\, dx = \frac{1}{4}x \Big|_{x=3}^{x=4} = (1) - \left(\frac{3}{4}\right) = \frac{1}{4}$$



Continuous Uniform Dist. (3/4)

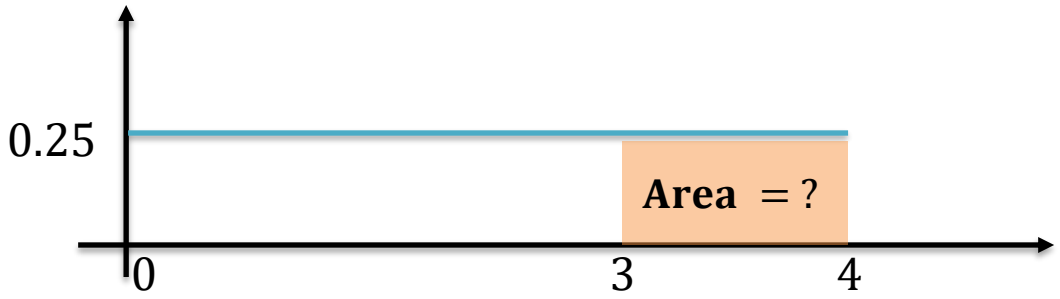
Example1 (5/6):

$\boldsymbol{X \sim U(0,4)}$

Distribution $F(x) = \begin{cases} 0 & \text{if } x < a \\ \dfrac{x-a}{b-a} & \text{if } a \le x \le b \\ 1 & \text{if } b < x \end{cases}$

b) What is the probability that any given conference lasts at least 3 hours?

$$P(X \geq 3) = 1 - P(X < 3) = 1 - F(3)$$



Continuous Uniform Dist. (3/4)

Example1 (6/6):

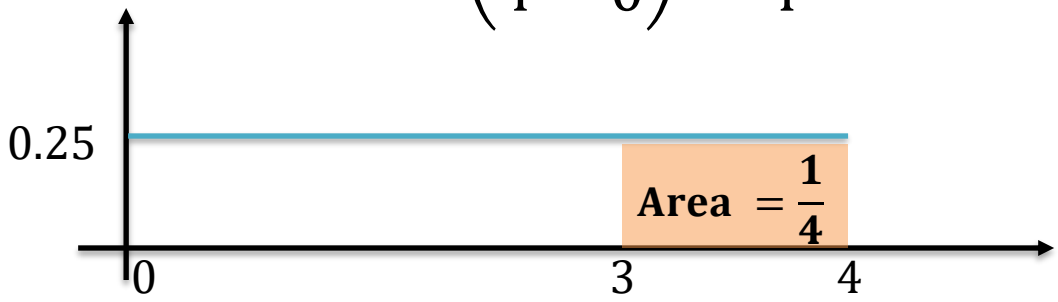
$\boldsymbol{X \sim U(0,4)}$

Distribution $$F(x) = \begin{cases} 0 & \text{if } x < a \\ \dfrac{x-a}{b-a} & \text{if } a \le x \le b \\ 1 & \text{if } b < x \end{cases}$$

b) What is the probability that any given conference lasts at least 3 hours?

$$P(X \ge 3) = 1 - P(X < 3) = 1 - F(3)$$

$$= 1 - \left(\frac{3-0}{4-0}\right) = \frac{1}{4}$$



Continuous Uniform Dist. (4/4)

Example2 (1/10):

A bus arrives every 10 minutes at a bus stop. It is assumed that the waiting time for a particular individual is a random variable with a continuous uniform distribution.

(a) What is the probability that the individual waits at most 4 minutes?

(b) What is the probability that the individual waits more than 6 minutes (at least 6 minutes)?

(c) What is the probability that the individual waits between 1 and 4 minutes?

(d) What is the average waiting time?

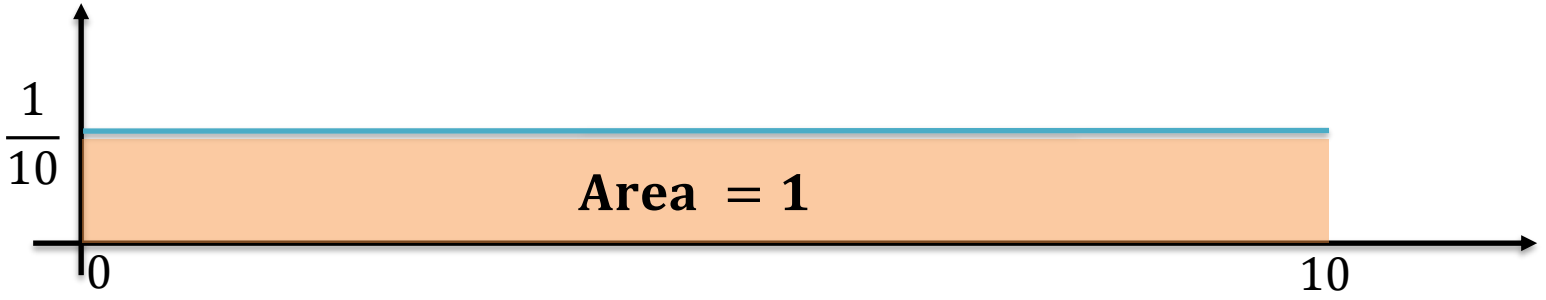
Continuous Uniform Dist. (4/4)

Example2 (2/10):

$X \sim U(0, 10)$

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{if } a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

$$f(x) = \begin{cases} \frac{1}{10} & \text{if } 0 \le x \le 10 \\ 0 & \text{otherwise} \end{cases}$$



Continuous Uniform Dist. (4/4)

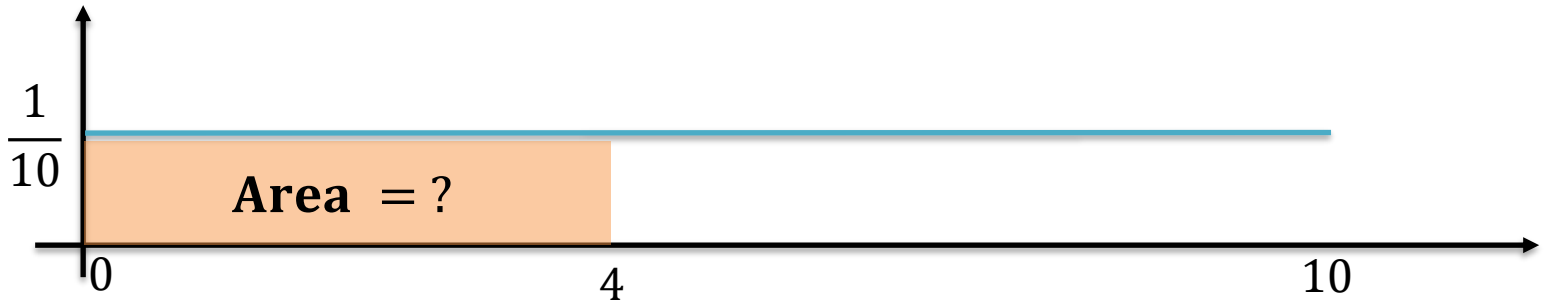
Example2 (3/10):

$\boldsymbol{X \sim U(0, 10)}$

Distribution $$F(x) = \begin{cases} 0 & \text{if } x < a \\ \dfrac{x-a}{b-a} & \text{if } a \le x \le b \\ 1 & \text{if } b < x \end{cases}$$

a) What is the probability that the individual waits at most 4 minutes?

$$P(X \le 4) = F(4)$$



Continuous Uniform Dist. (4/4)

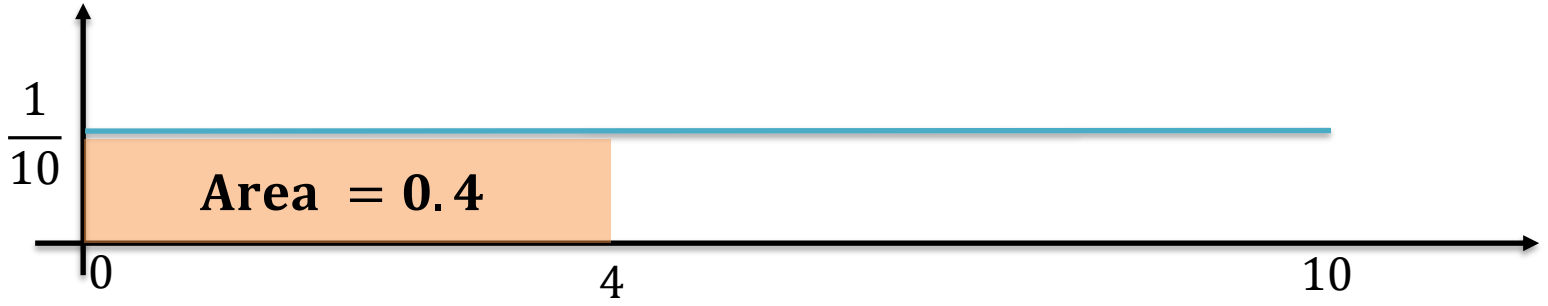
Example2 (4/10):

$\boldsymbol{X \sim U(0, 10)}$

Distribution $$F(x) = \begin{cases} 0 & \text{if } x < a \\ \dfrac{x-a}{b-a} & \text{if } a \le x \le b \\ 1 & \text{if } b < x \end{cases}$$

a) What is the probability that the individual waits at most 4 minutes?

$$P(X \le 4) = F(4) = \frac{4-0}{10-0} = \frac{4}{10} = 0.4$$



Continuous Uniform Dist. (4/4)

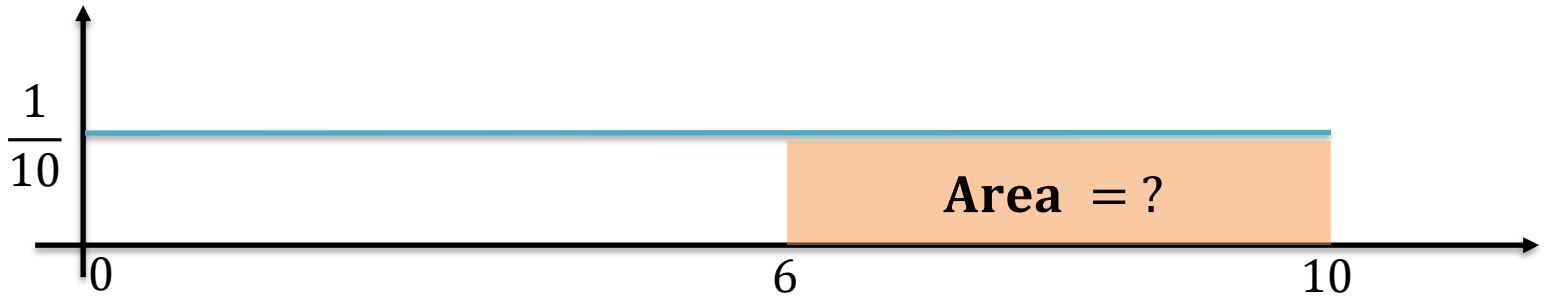
Example2 (5/10):

$\boldsymbol{X \sim U(0, 10)}$

Distribution $$F(x) = \begin{cases} 0 & \text{if } x < a \\ \dfrac{x-a}{b-a} & \text{if } a \le x \le b \\ 1 & \text{if } b < x \end{cases}$$

b) What is the probability that the individual waits more than 6 minutes (at least 6 minutes)?

$$P(X \geq 6) = 1 - F(6)$$



Continuous Uniform Dist. (4/4)

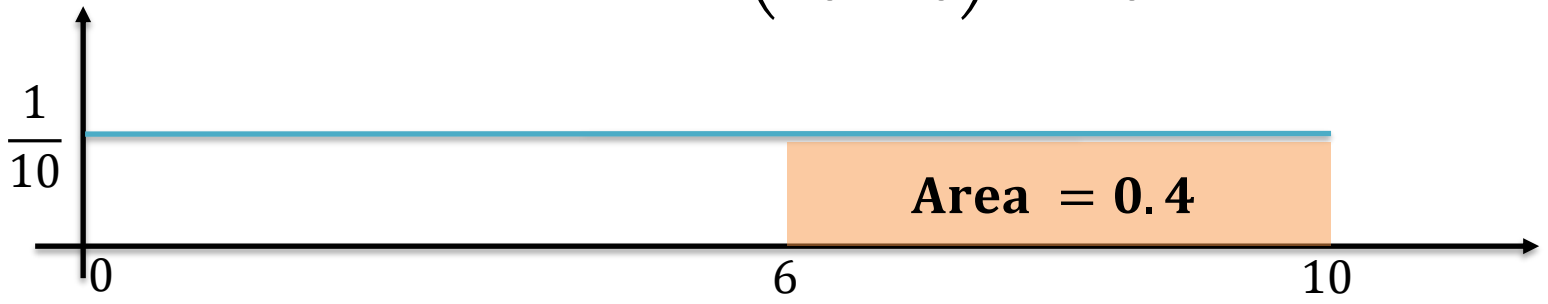
Example2 (6/10):

$\boldsymbol{X \sim U(0, 10)}$

Distribution $$F(x) = \begin{cases} 0 & \text{if } x < a \\ \dfrac{x-a}{b-a} & \text{if } a \le x \le b \\ 1 & \text{if } b < x \end{cases}$$

b) What is the probability that the individual waits more than 6 minutes (at least 6 minutes)?

$$P(X \geq 6) = 1 - \left(\frac{6-0}{10-0}\right) = \frac{4}{10} = 0.4$$



Continuous Uniform Dist. (4/4)

Example2 (7/10):

$\boldsymbol{X \sim U(0, 10)}$

c) What is the probability that the individual waits between 1 and 4 minutes?

$$P(1 \leq X \leq 4)$$

Continuous Uniform Dist. (4/4)

Example2 (8/10):

$\boldsymbol{X \sim U(0, 10)}$

c) What is the probability that the individual waits between 1 and 4 minutes?

$$P(1 \le X \le 4) = \int_1^4 \frac{1}{10}\, dx = \frac{1}{10} x \Big|_{x=1}^{x=4} = \left(\frac{4}{10}\right) - \left(\frac{1}{10}\right) = \frac{3}{10}$$

Continuous Uniform Dist. (4/4)

Example2 (9/10):

$\boldsymbol{X \sim U(0, 10)}$

Distribution $$F(x) = \begin{cases} 0 & \text{if } x < a \\ \dfrac{x-a}{b-a} & \text{if } a \le x \le b \\ 1 & \text{if } b < x \end{cases}$$

c) What is the probability that the individual waits between 1 and 4 minutes?

$$P(1 \le X \le 4) = F(4) - F(1) = \left(\frac{4-0}{10-0}\right) - \left(\frac{1-0}{10-0}\right) = \frac{3}{10}$$

Continuous Uniform Dist. (4/4)

Example2 (10/10):

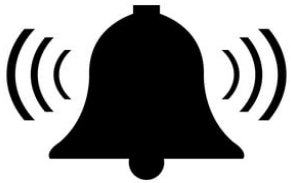
$$\boldsymbol{X \sim U(0, 10)}$$

$$\mu = E(X) = \frac{a + b}{2}$$

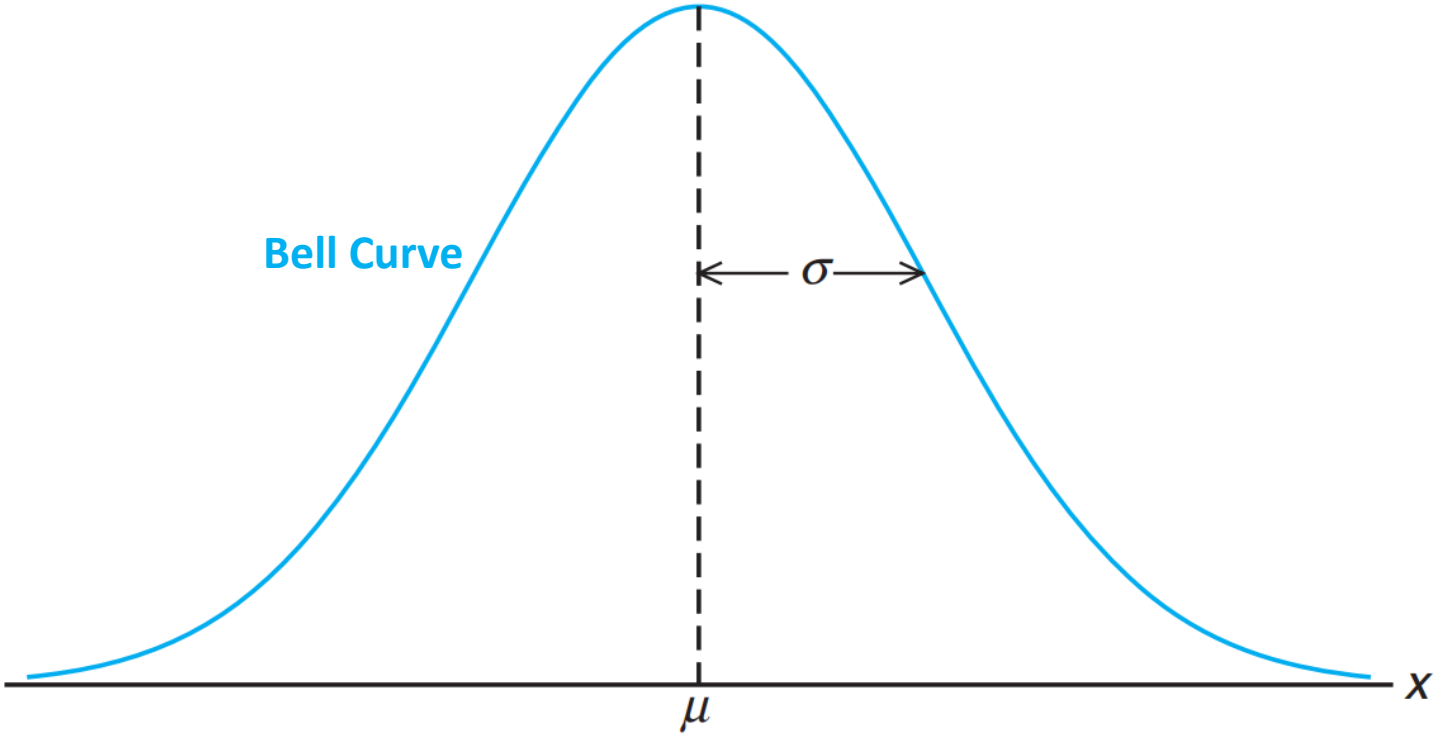
d) What is the average waiting time?

$$\mu = E(X) = \frac{0 + 10}{2} = 5 \text{ minutes}$$

Normal Distribution (1/41)



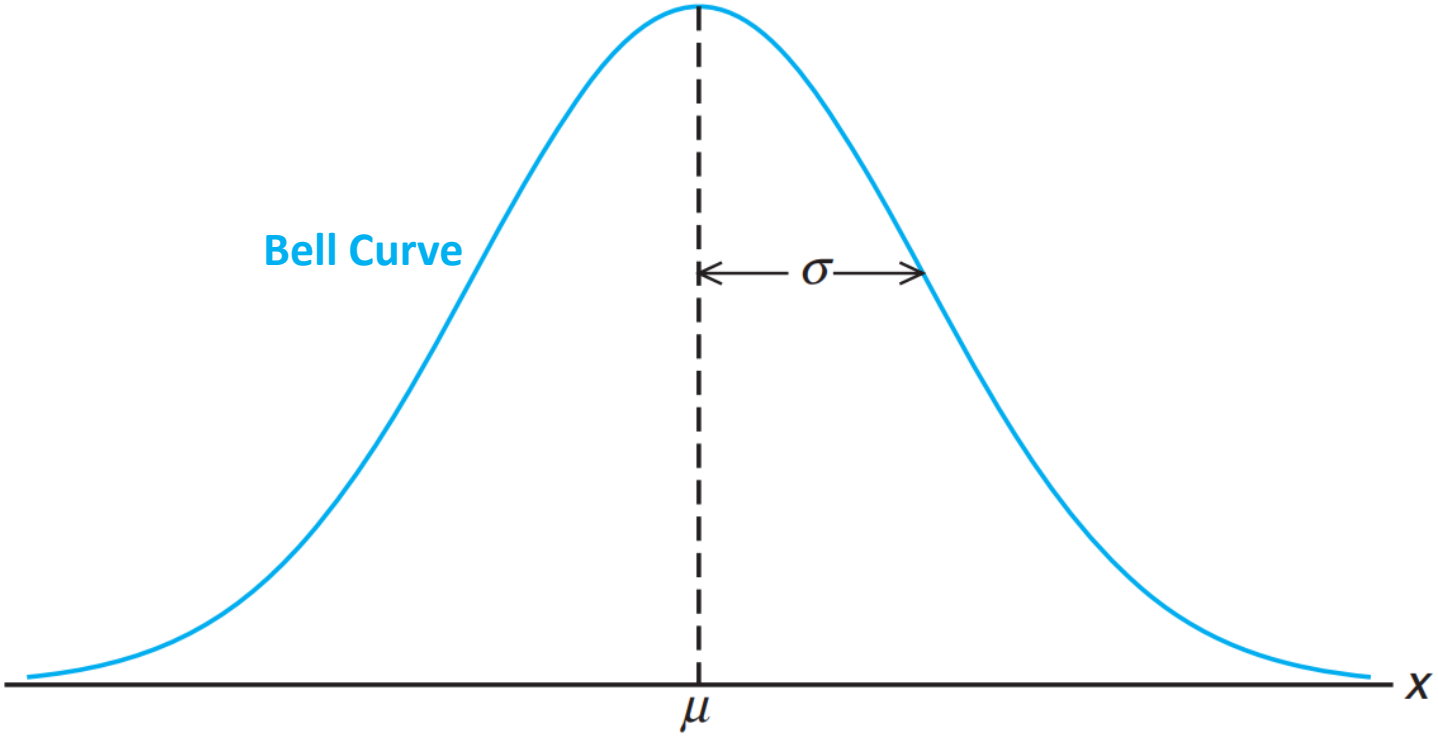
Bell



Normal Distribution (2/41)

A continuous random variable X having the bell-shaped distribution of the following figure is called a *normal random variable*.

Its graph, called the *normal curve*.



Normal Distribution (2/41)

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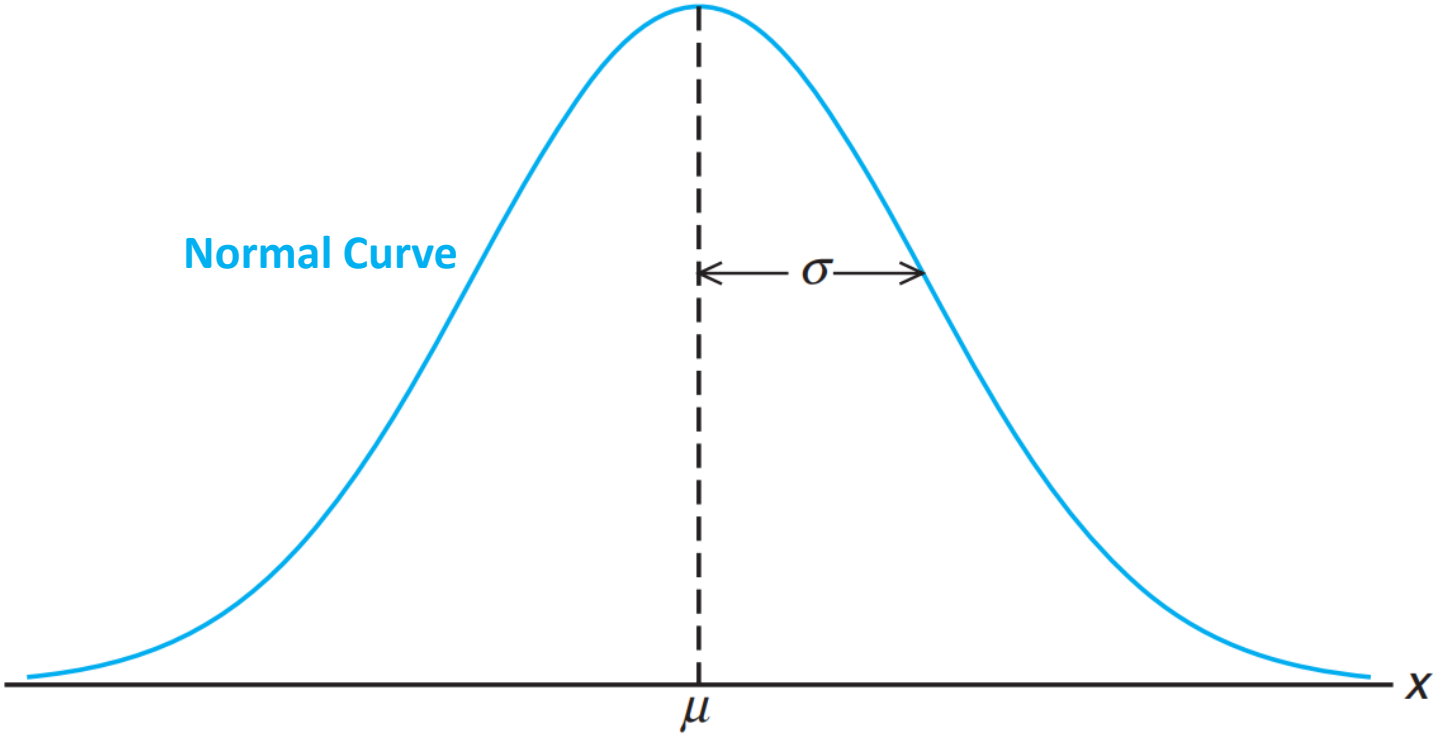
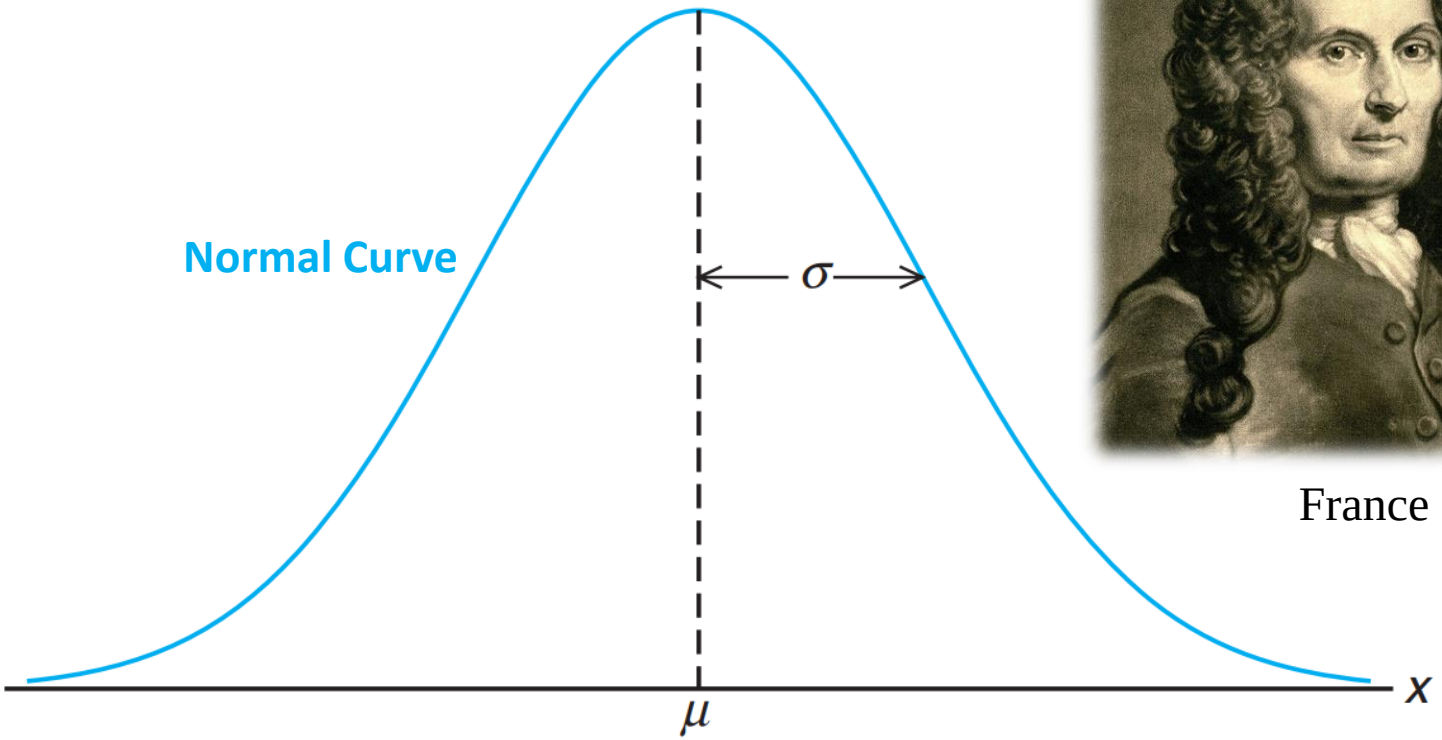
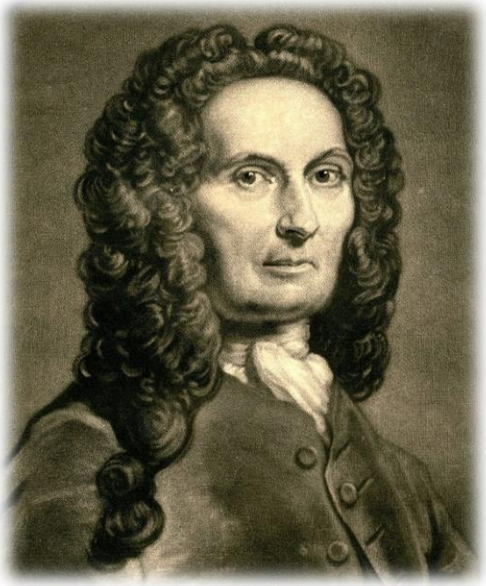


Normal Distribution (3/41)

In 1733, **Abraham DeMoivre** developed the mathematical equation of the normal curve.

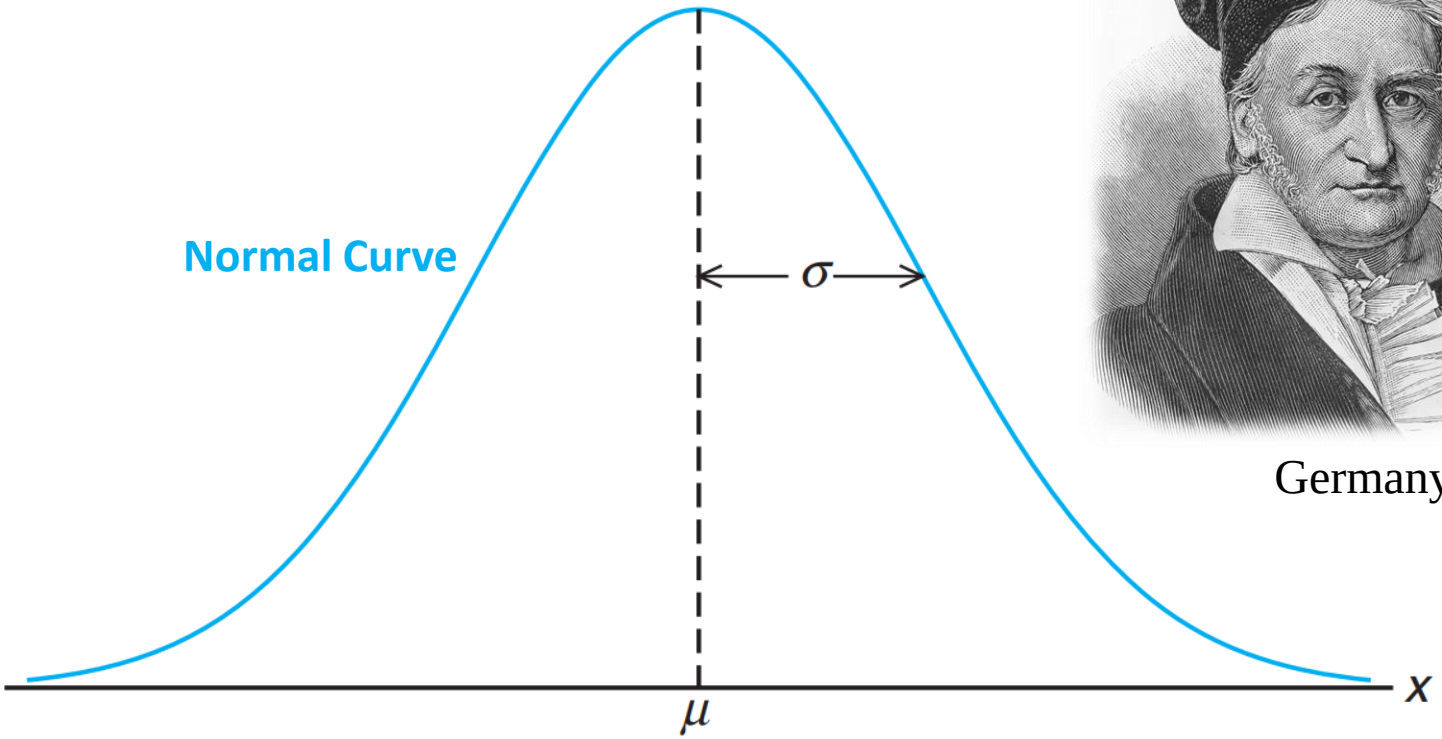




France

Normal Distribution (4/41)

The normal distribution is often referred to as the **Gaussian distribution**, in honor of Karl Friedrich Gauss, who discovered the normal distribution in 1823.



Germany

Normal Distribution (5/41)

X : continuous random variable

μ : mean

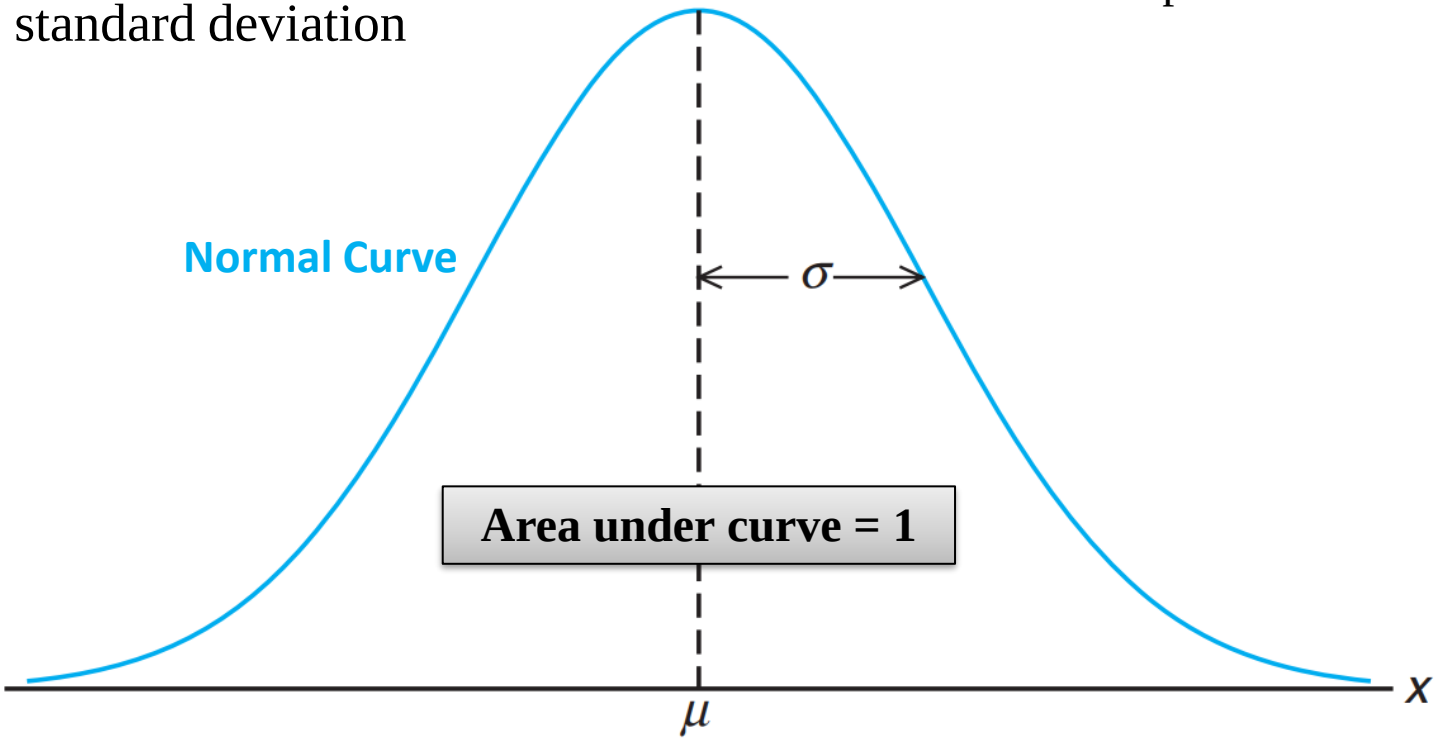
σ^2: variance

σ : standard deviation

$$X \sim N(\mu, \sigma^2)$$

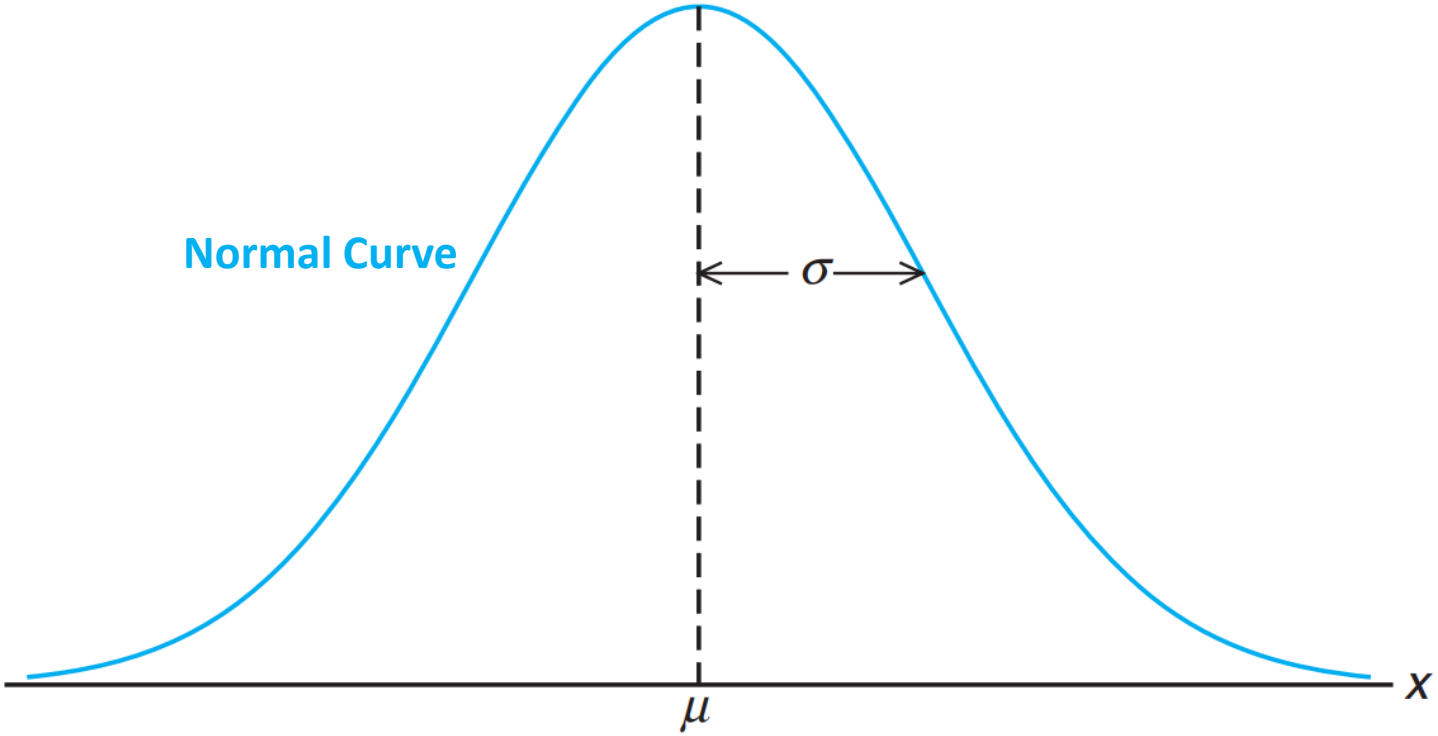
Location parameter $\mu \in \mathbb{R}$,

Scale parameter $\sigma > 0$.



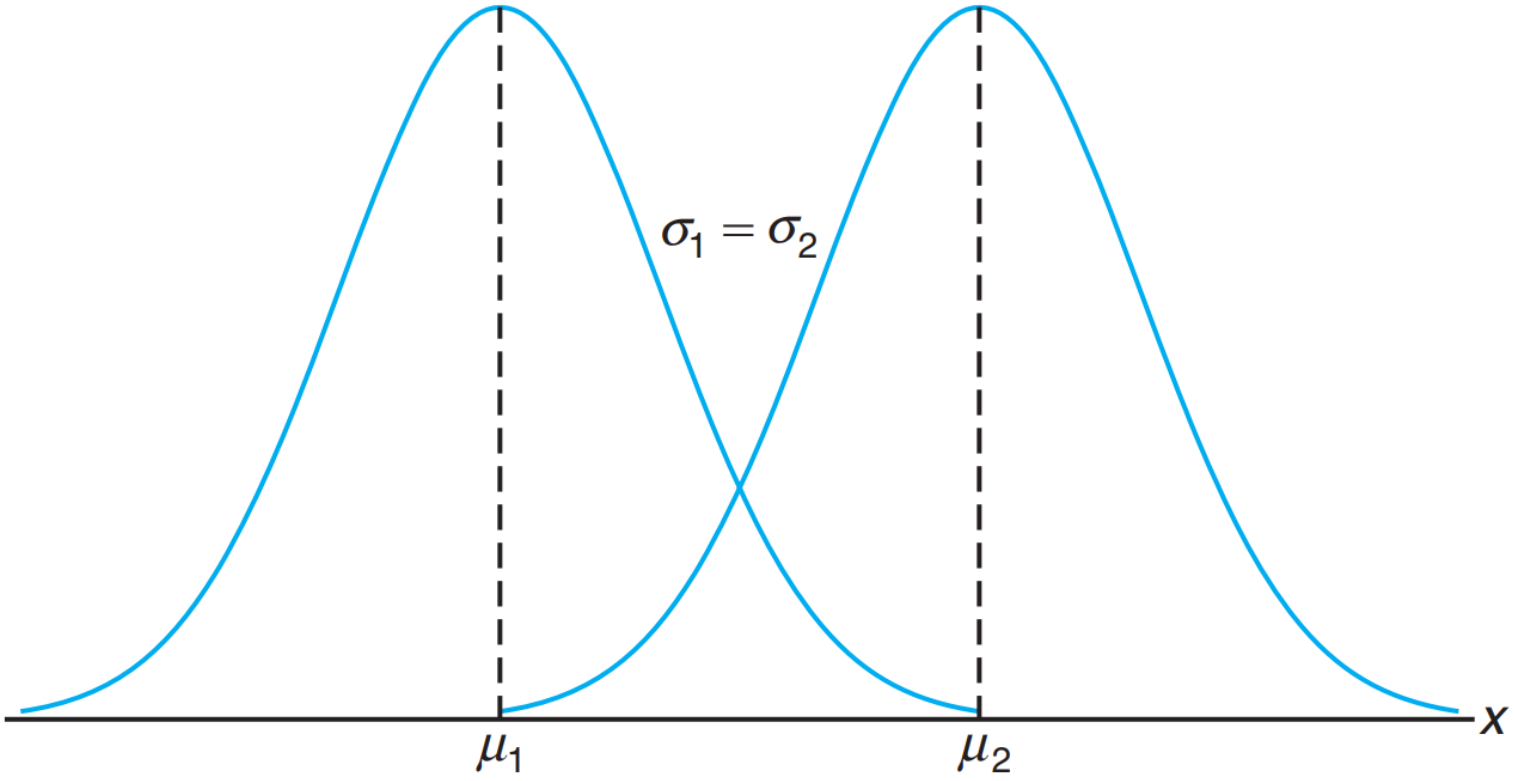
Normal Distribution (6/41)

Note that the graph of the normal distribution is ***symmetric*** about its mean μ (i.e., for any $c > 0$, $P(X > \mu + c) = P(X < \mu - c)$, and that σ^2 is a measure of the width of the bell shape.



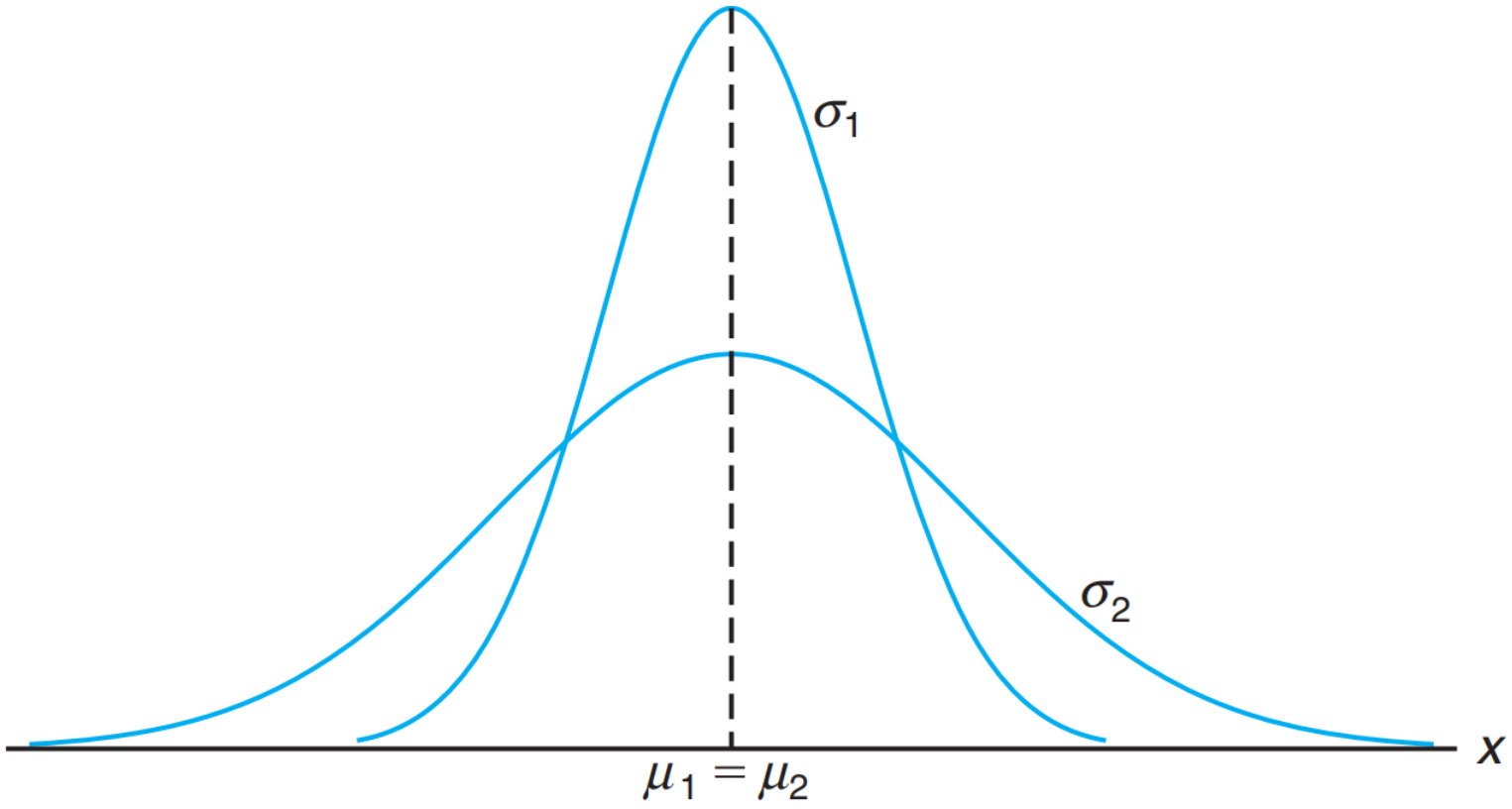
Normal Distribution (7/41)

Same standard deviation but different means $(\mu_1 < \mu_2)$.



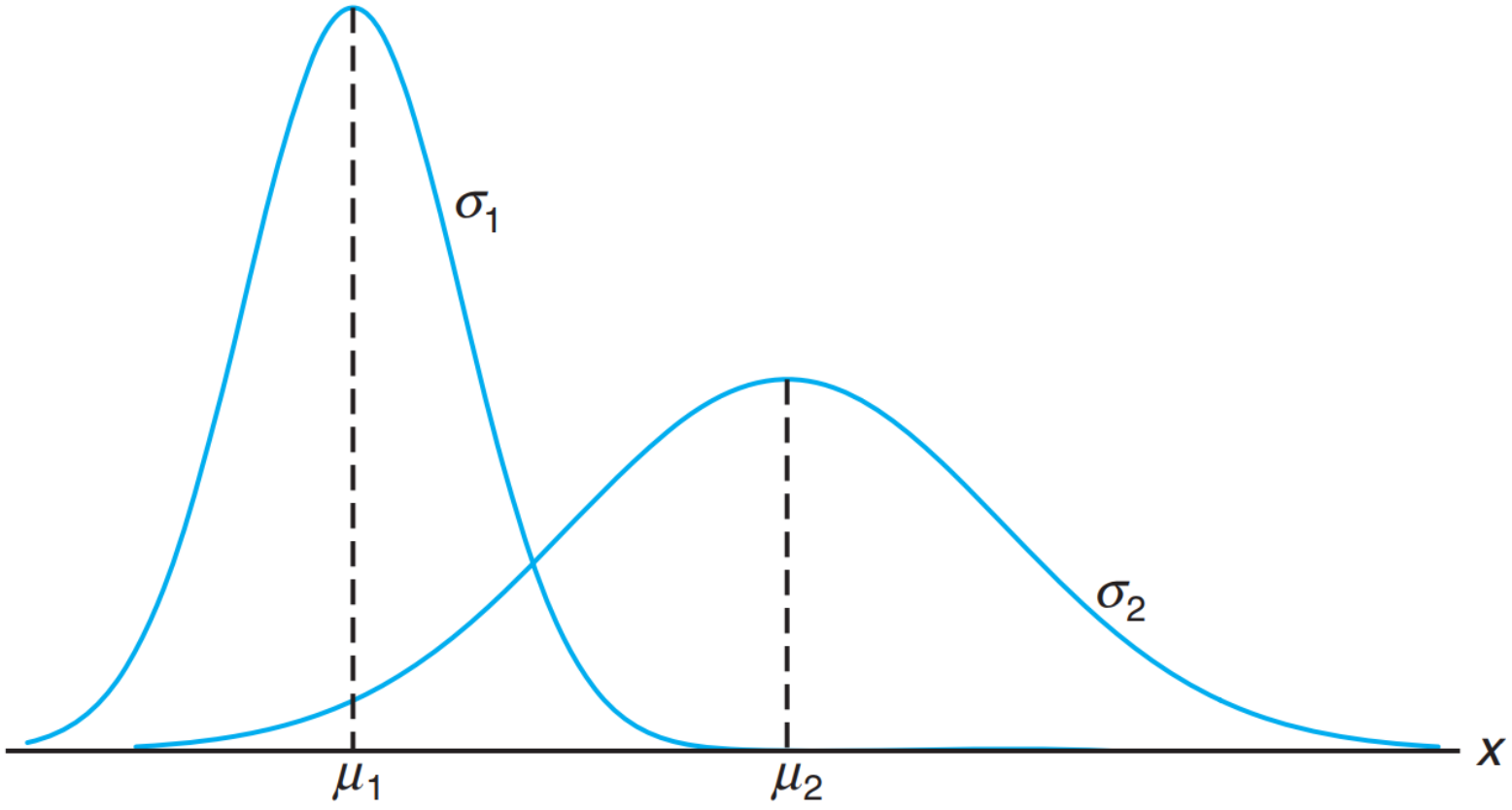
Normal Distribution (8/41)

Same mean but different standard deviations ($\sigma_1 < \sigma_2$).



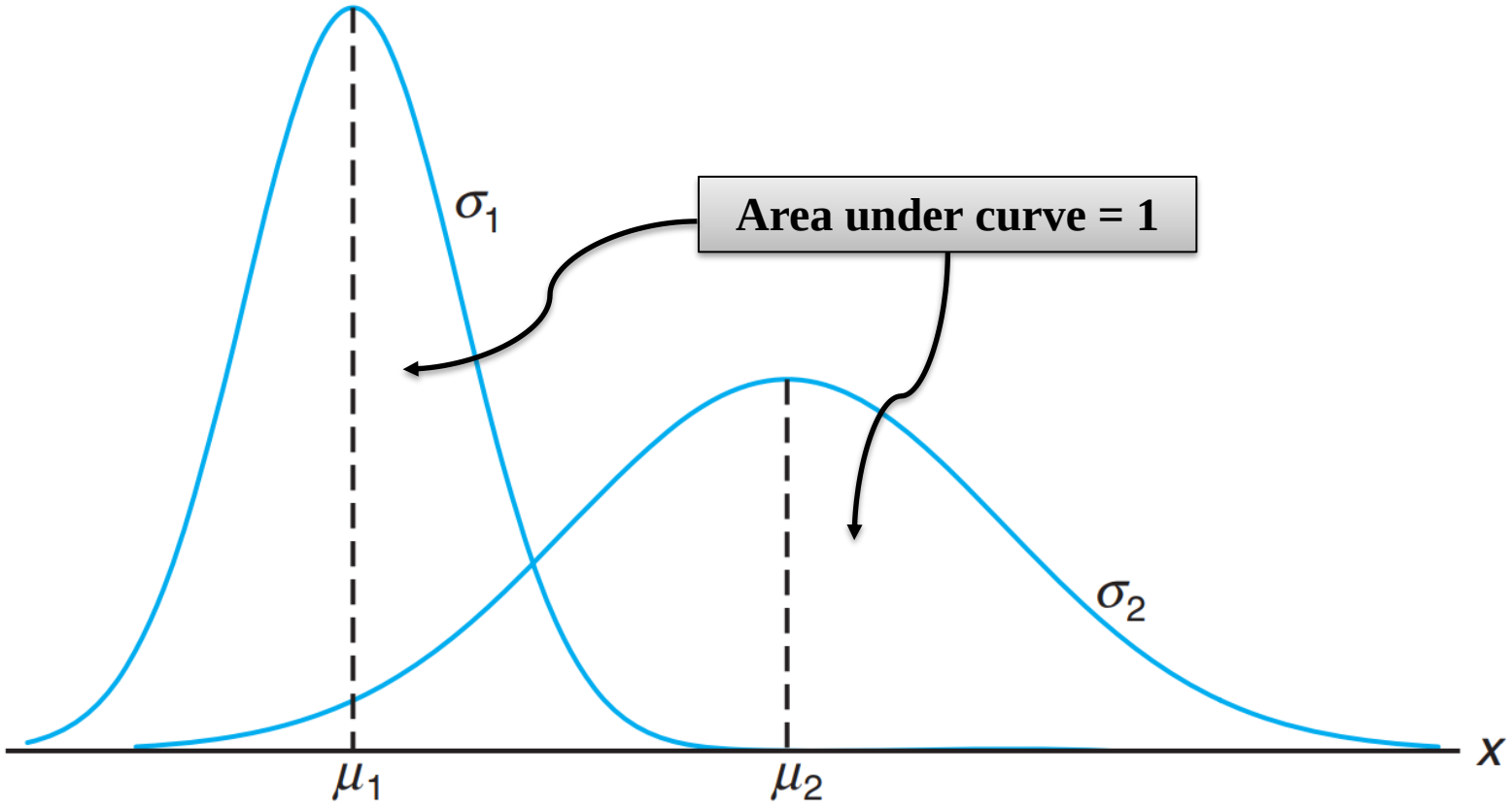
Normal Distribution (9/41)

Different means $(\mu_1 < \mu_2)$ and standard deviations $(\sigma_1 < \sigma_2)$.



Normal Distribution (10/41)

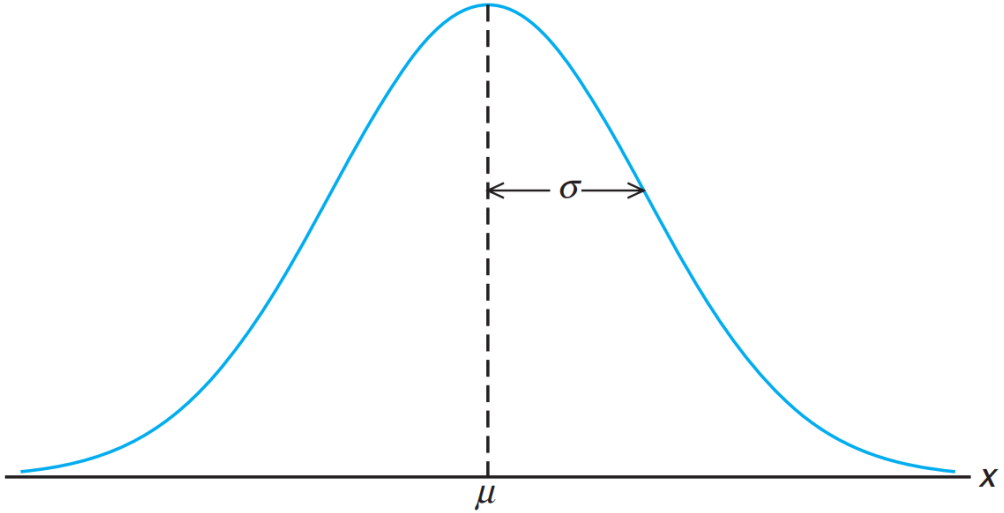
Different means $(\mu_1 < \mu_2)$ and standard deviations $(\sigma_1 < \sigma_2)$.



Normal Distribution (11/41)

Properties of the normal curve (1/2):

1) Mode = Mean (μ)
2) The curve is symmetric about a vertical axis through the mean (μ).
3) The curve has its points of inflection at $x = \mu \pm \sigma$; it is concave downward if $\mu - \sigma < X < \mu + \sigma$ and is concave upward otherwise.

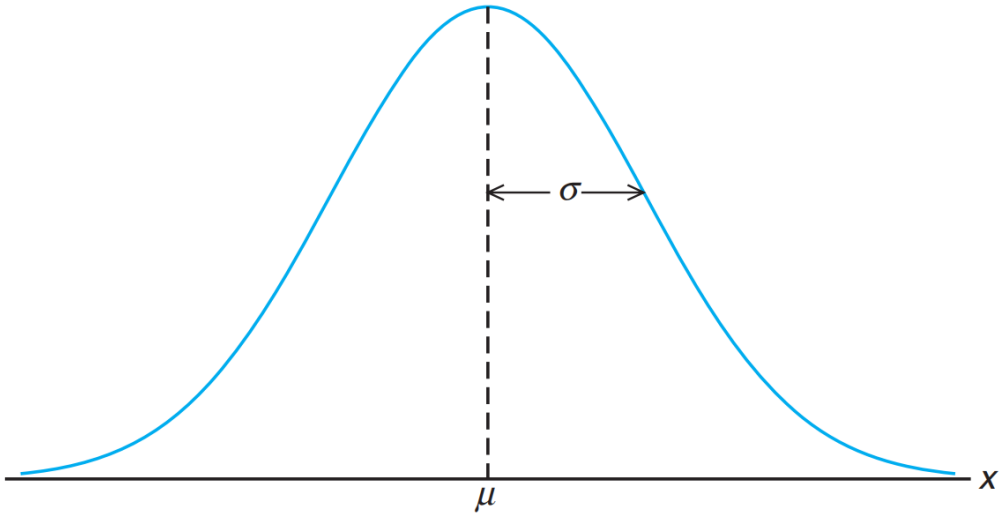


Normal Distribution (11/41)

Properties of the normal curve (2/2):

4) The normal curve approaches the horizontal axis asymptotically as we proceed in either direction away from the mean.

5) The total area under the curve and above the horizontal axis is equal to 1.



Normal Distribution (12/41)

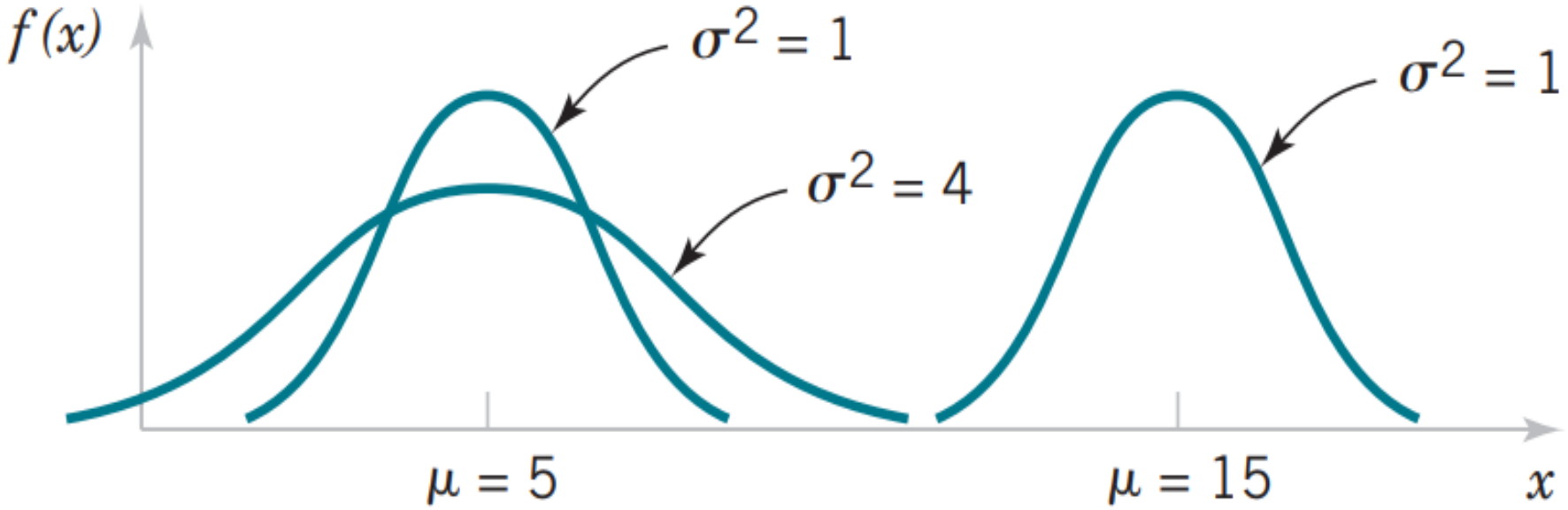
Probability density function of the normal random variable X:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}} \qquad -\infty < x < \infty$$

The mean and variance of X are μ and σ^2, respectively,

where $-\infty < \mu < \infty$ and $\sigma > 0$

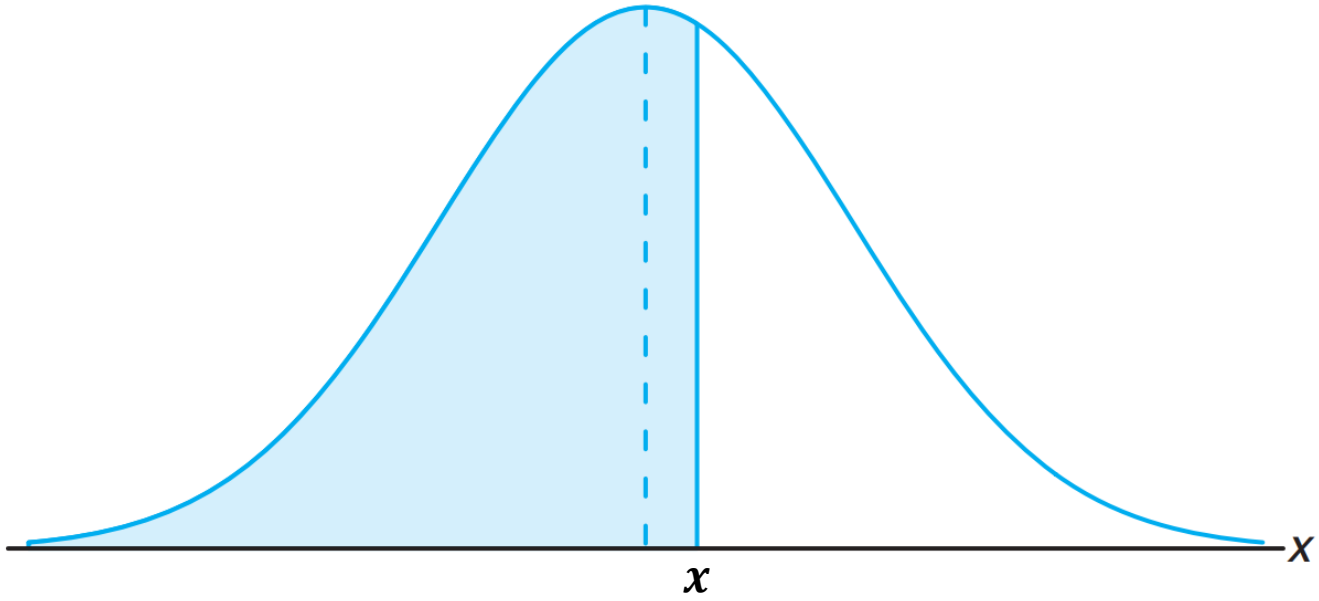
Normal Distribution (13/41)



Normal Distribution (14/41)

Cumulative distribution function:

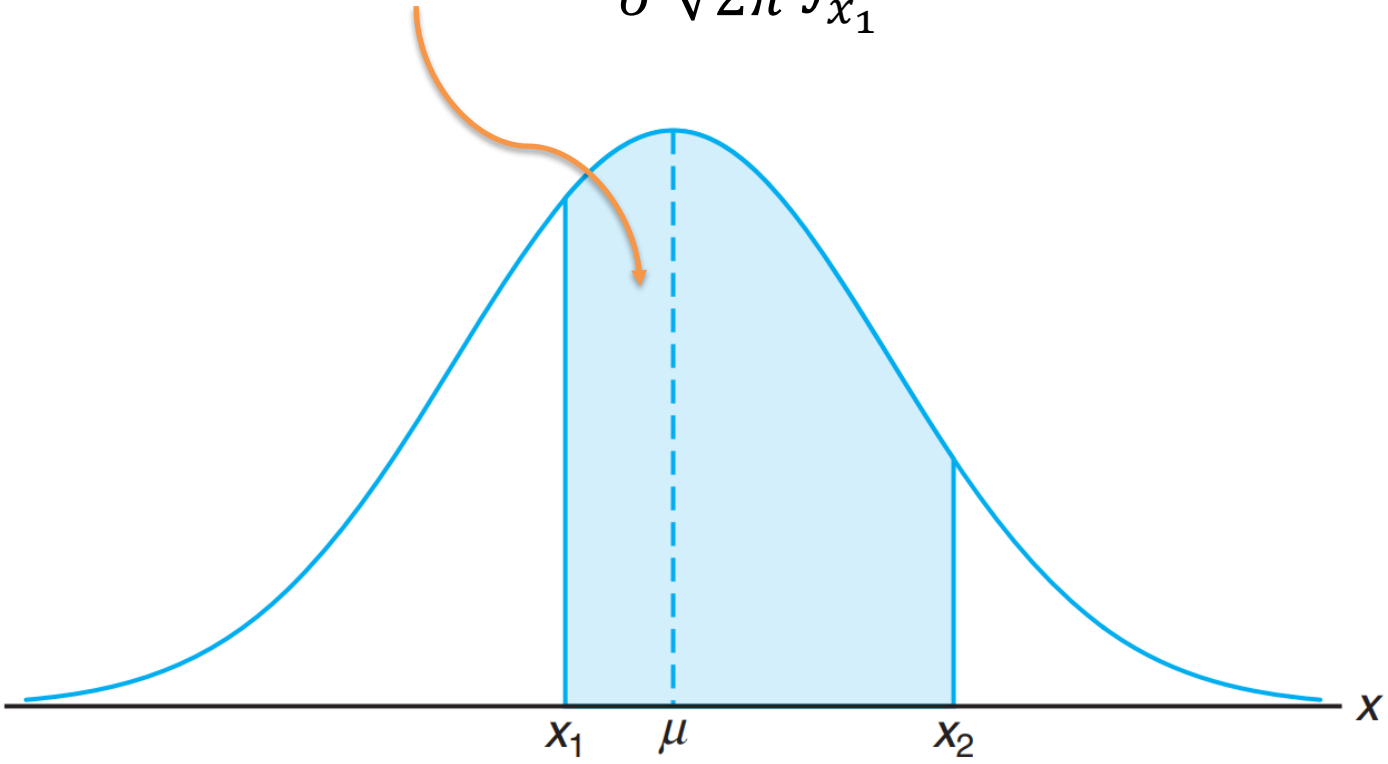
$$F(x) = P(X \leq x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} e^{\frac{-(v-\mu)^2}{2\sigma^2}} \, dv$$



Normal Distribution (15/41)

Areas under the Normal Curve:

$$P(x_1 < X < x_2) = \frac{1}{\sigma\sqrt{2\pi}} \int_{x_1}^{x_2} e^{\frac{-(x-\mu)^2}{2\sigma^2}} \, dx$$

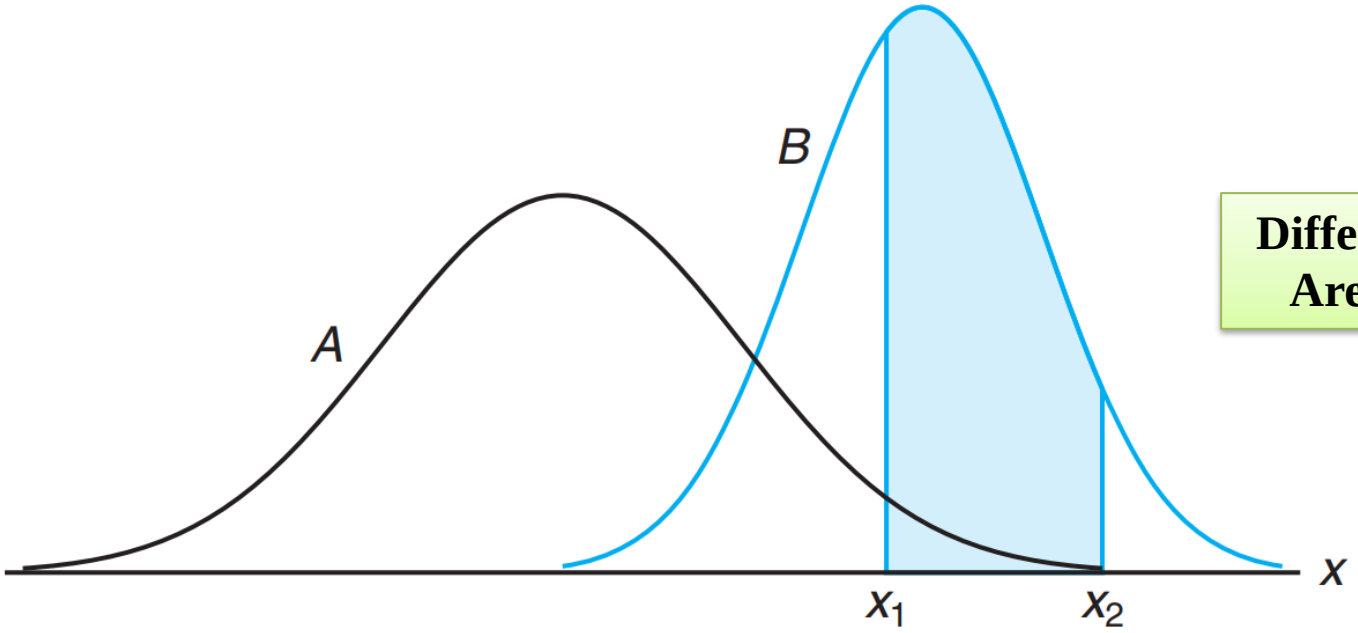


Normal Distribution (16/41)

Areas under the Normal Curve:

Different means and standard deviations

$$P(x_1 < X < x_2) = \frac{1}{\sigma\sqrt{2\pi}} \int_{x_1}^{x_2} e^{\frac{-(x-\mu)^2}{2\sigma^2}} \, dx$$



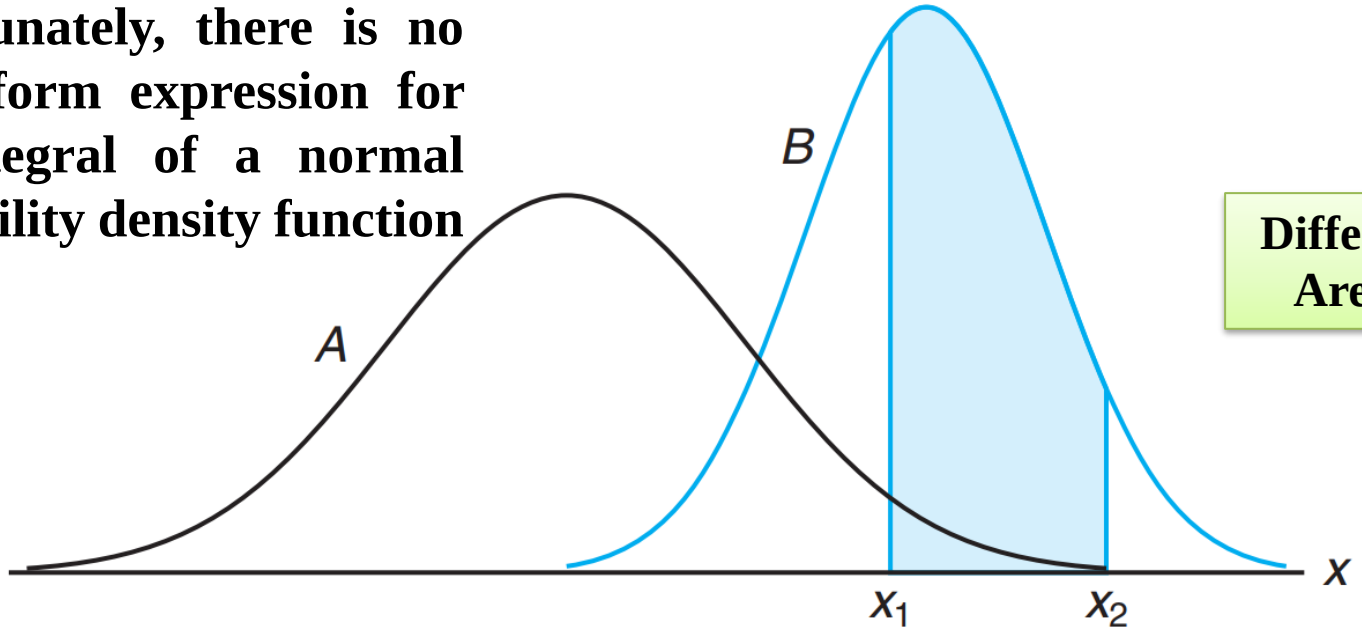
Normal Distribution (16/41)

Areas under the Normal Curve:

Different means and standard deviations

$$P(x_1 < X < x_2) = \frac{1}{\sigma\sqrt{2\pi}} \int_{x_1}^{x_2} e^{\frac{-(x-\mu)^2}{2\sigma^2}} \, dx$$

Unfortunately, there is no closed-form expression for the integral of a normal probability density function

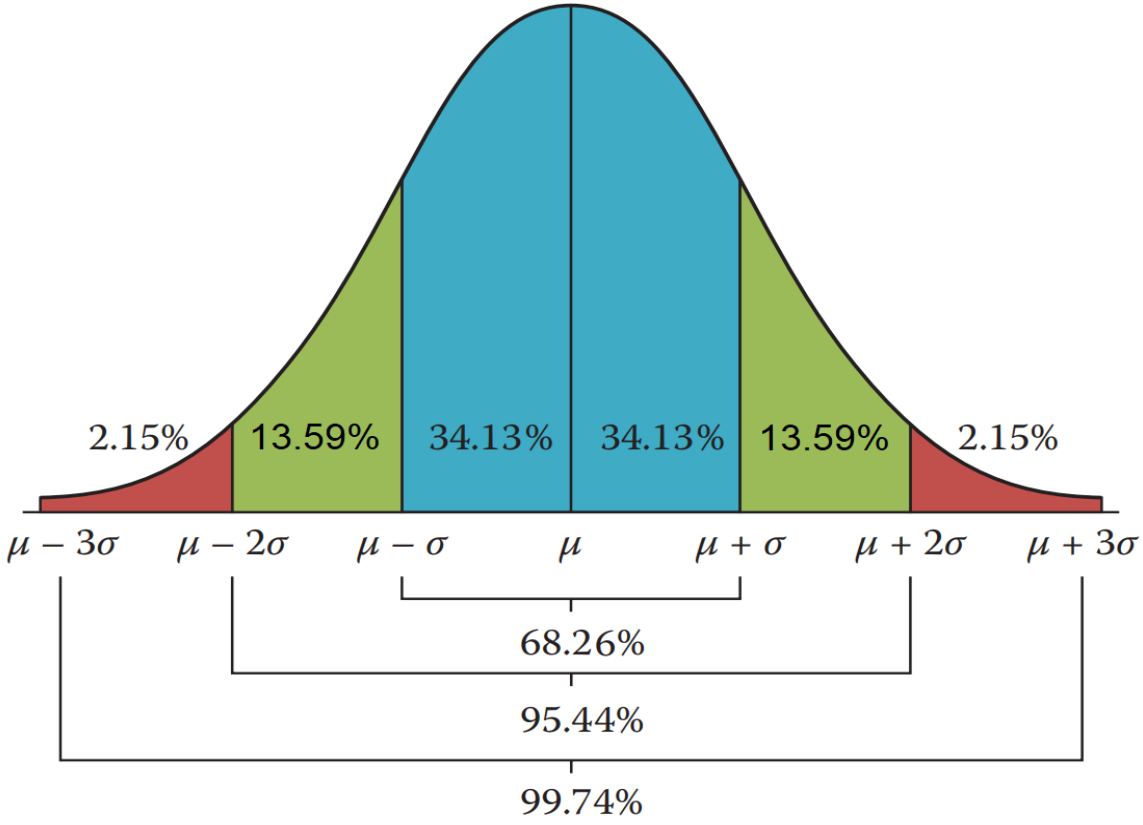


Different Areas

Normal Distribution (17/41)

Empirical Rule (1/2):

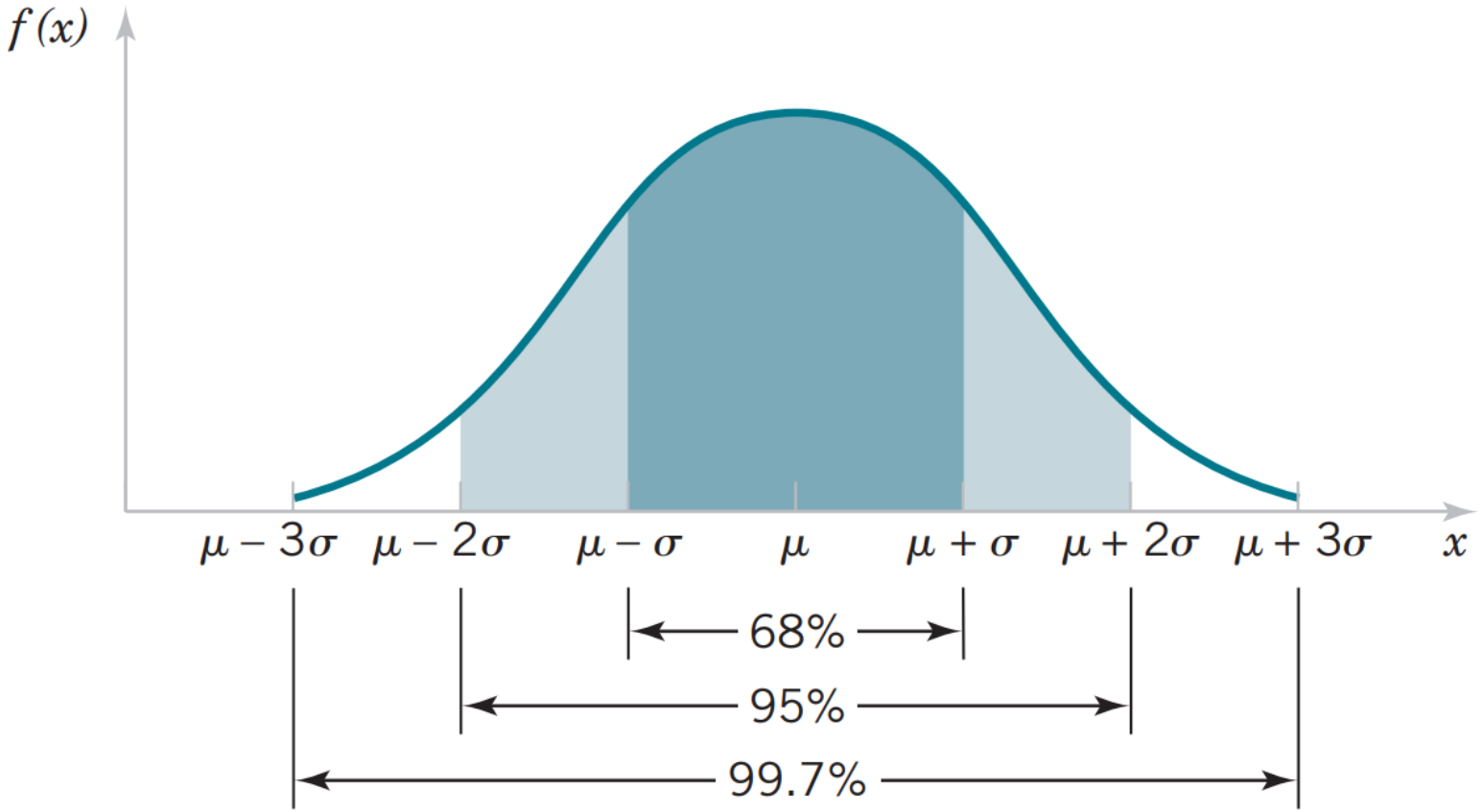
68-95-99.7 rule *or* three-sigma rule



Normal Distribution (17/41)

Empirical Rule (2/2):

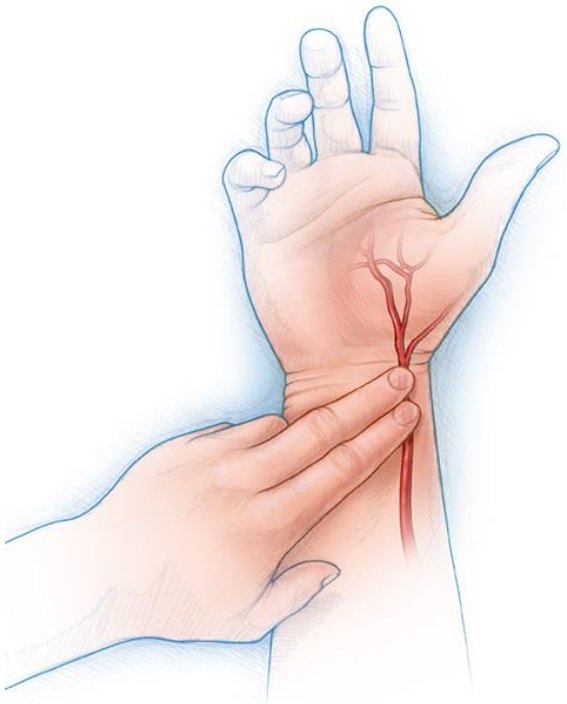
68-95-99.7 rule *or* three-sigma rule



Normal Distribution (18/41)

Example1 (1/5):

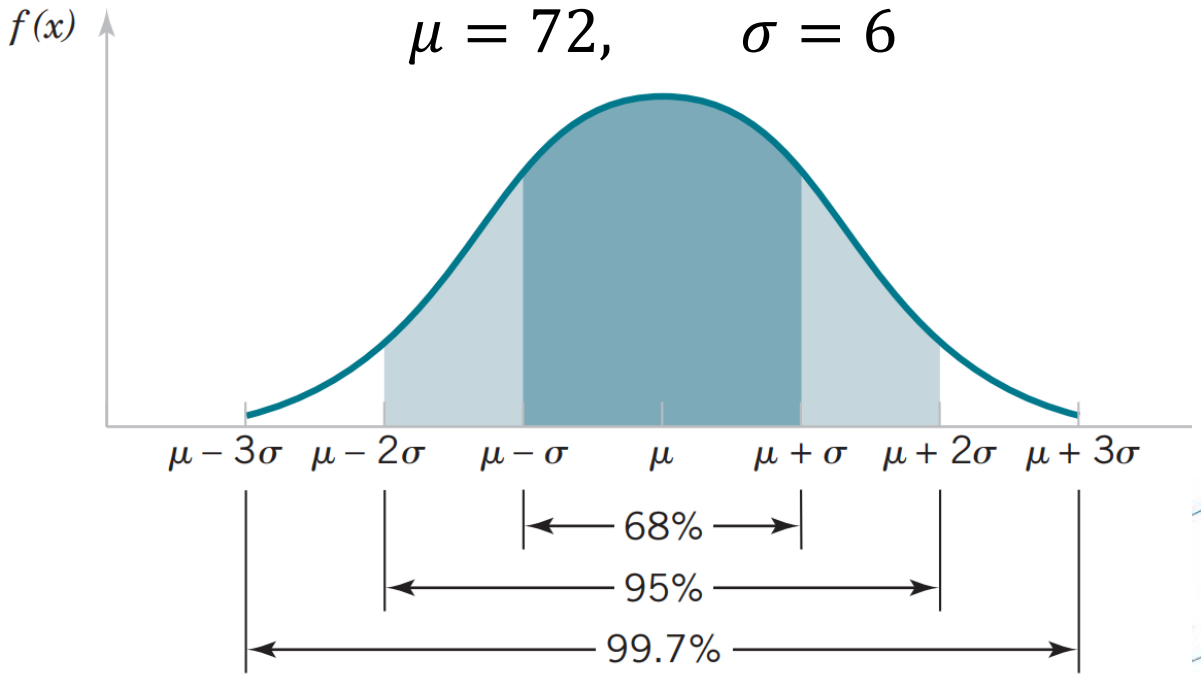
Suppose the pulse rates of 200 college men are bell-shaped with a mean of 72 beats per minute (bpm) and a standard deviation of 6 bpm.

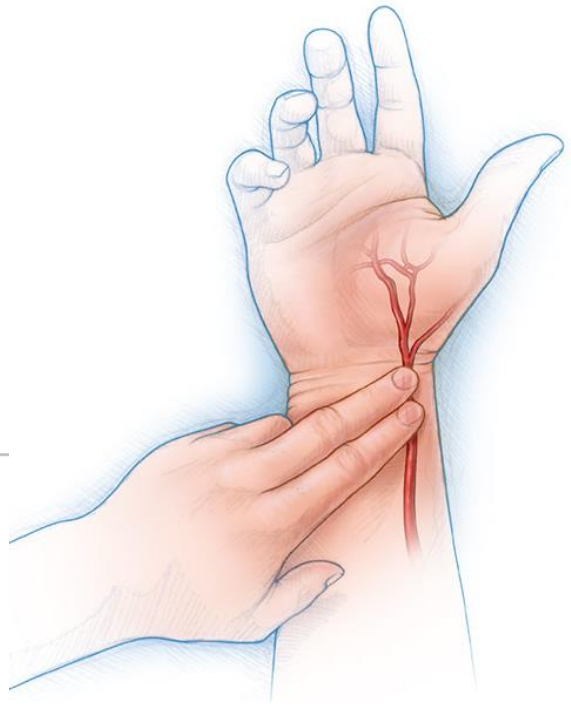


Normal Distribution (18/41)

Example1 (2/5):

Suppose the pulse rates of 200 college men are bell-shaped with a mean of 72 bpm and standard deviation of 6 bpm.

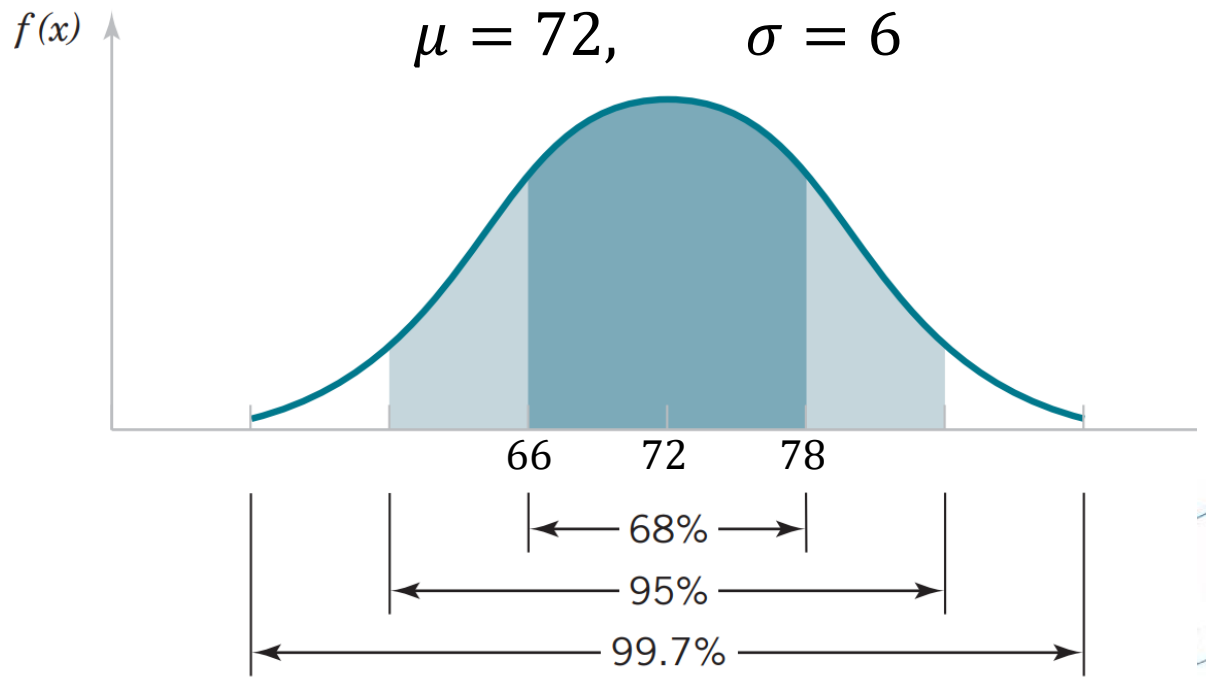


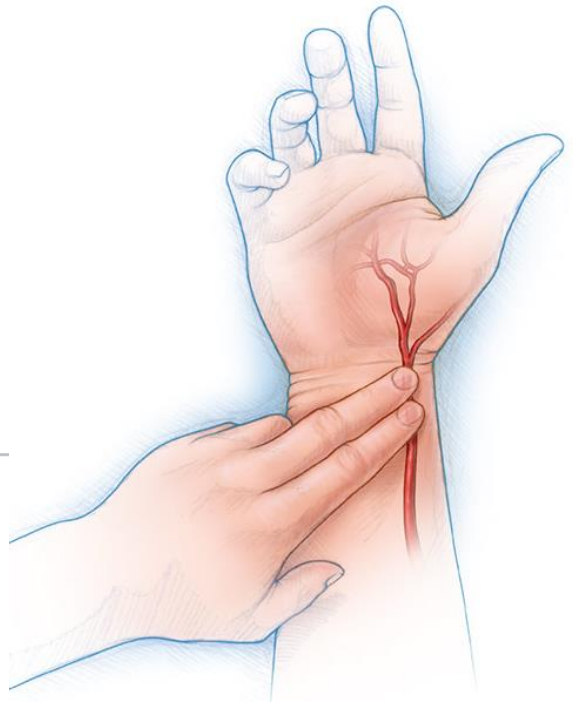


Normal Distribution (18/41)

Example1 (3/5):

About 68% of the men have pulse rates in the interval $[66, 78]$ bpm.

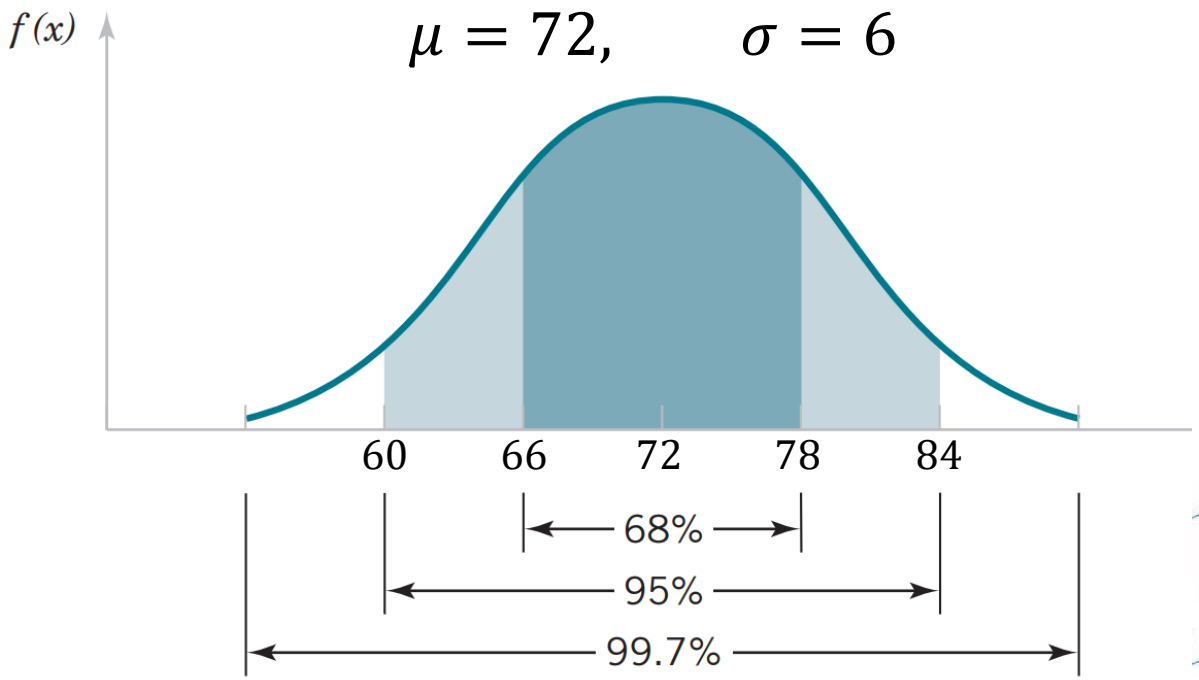


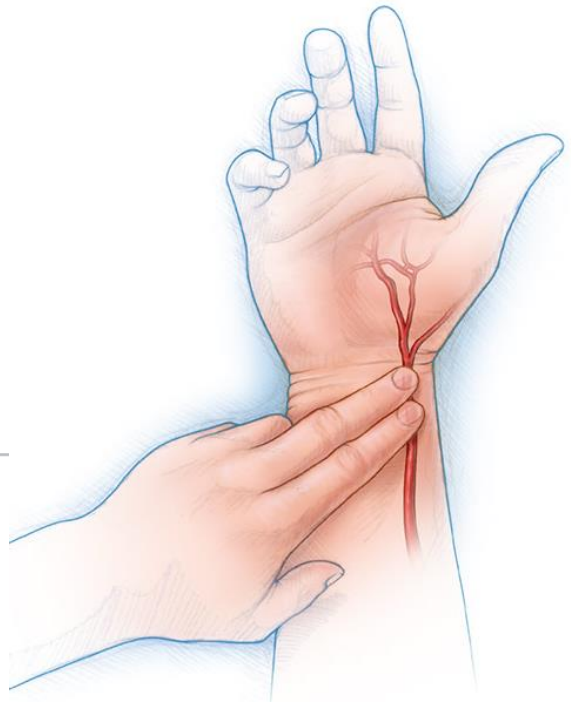


Normal Distribution (18/41)

Example1 (4/5):

About 95% of the men have pulse rates in the interval $[60, 84]$ bpm.

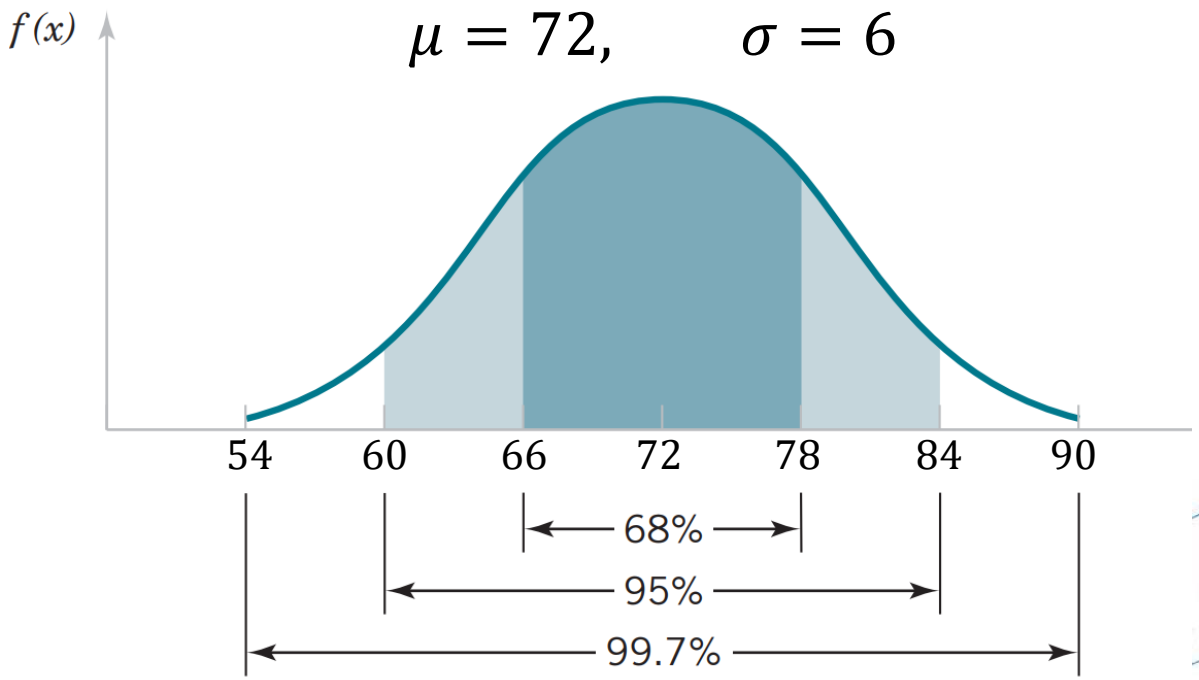


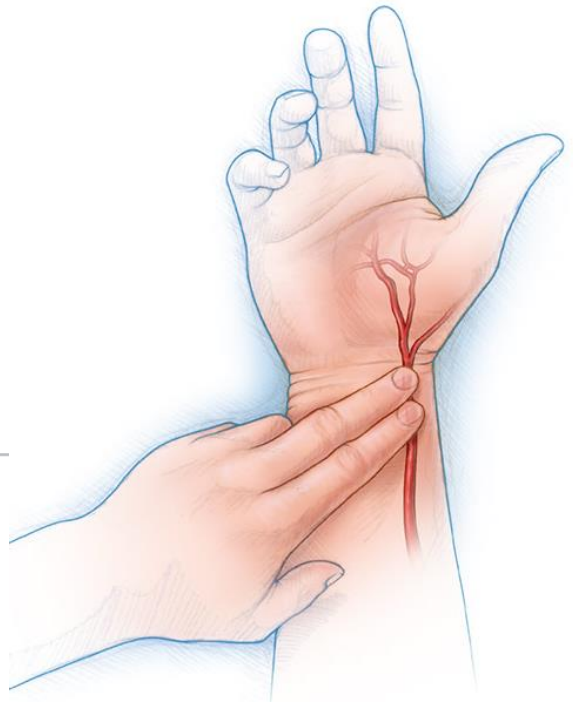


Normal Distribution (18/41)

Example1 (5/5):

About 99.7% of the men have pulse rates in the interval $[54, 90]$ bpm.



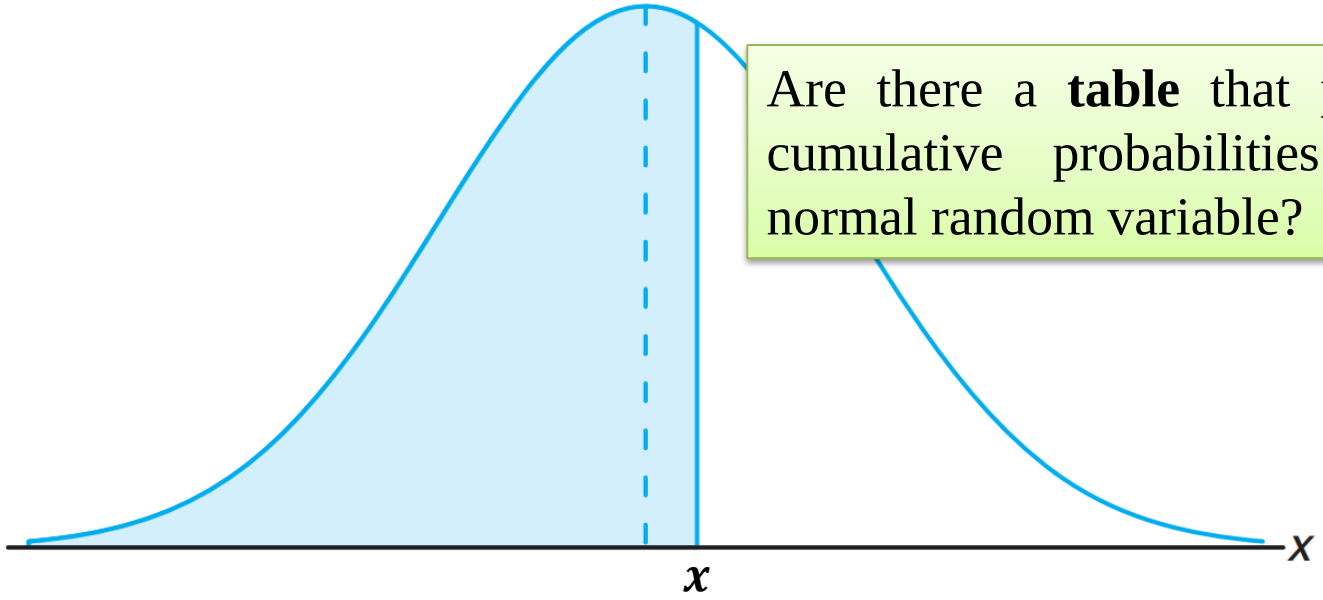


Normal Distribution (19/41)

Recall

Cumulative distribution function:

$$F(x) = P(X \leq x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} e^{\frac{-(v-\mu)^2}{2\sigma^2}} \, dv$$



Are there a **table** that provides cumulative probabilities for a normal random variable?

Normal Distribution (19/41)

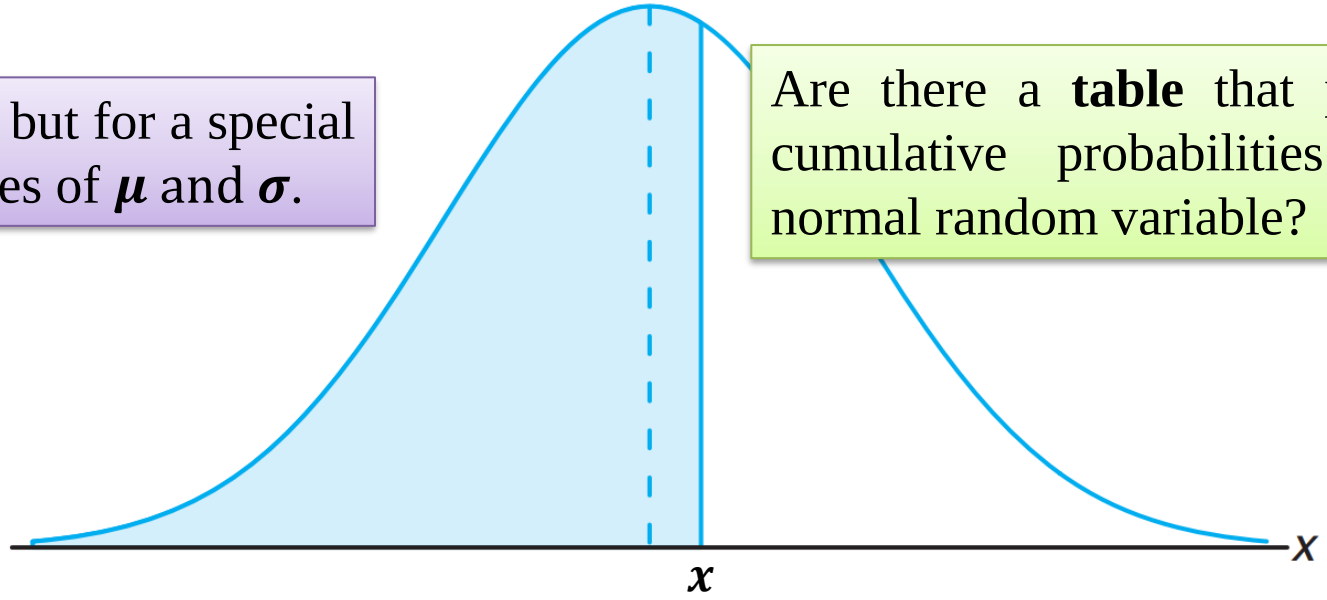
Recall

Cumulative distribution function:

$$F(x) = P(X \leq x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} e^{\frac{-(v-\mu)^2}{2\sigma^2}} \, dv$$

Yes, but for a special values of $\boldsymbol{\mu}$ and $\boldsymbol{\sigma}$.

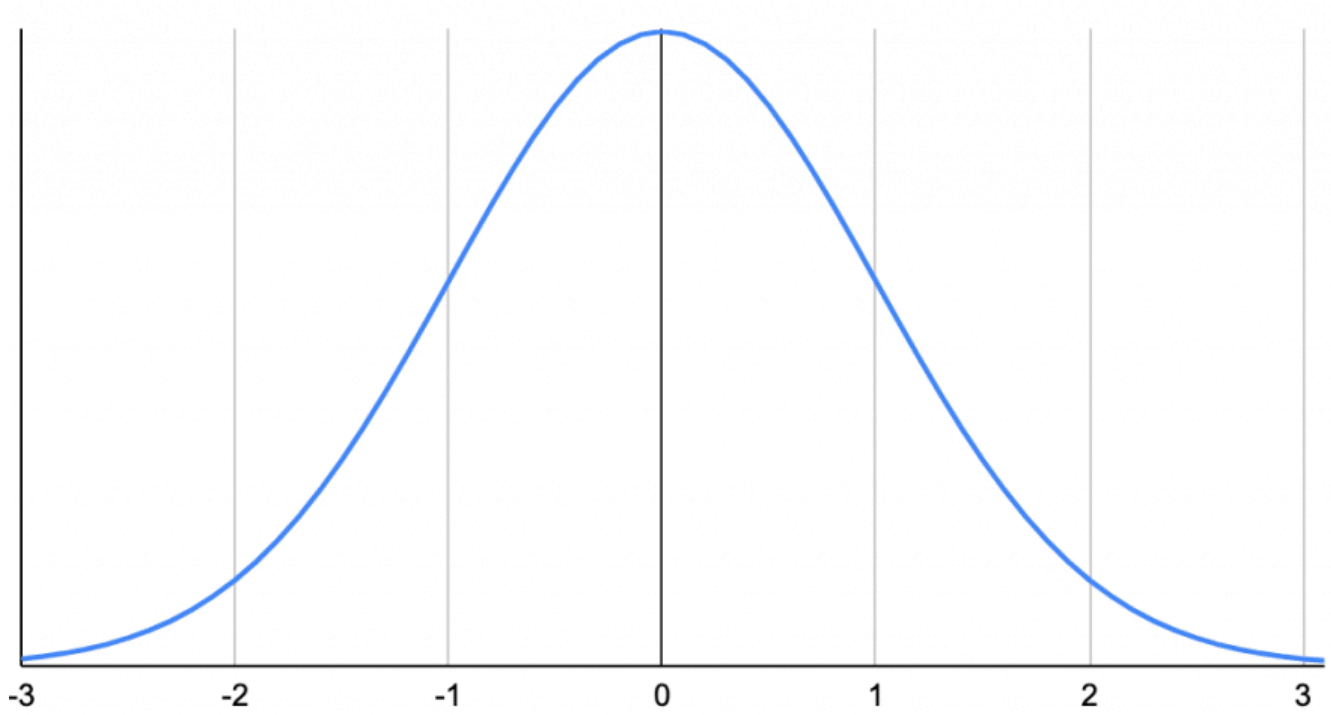
Are there a **table** that provides cumulative probabilities for a normal random variable?



Normal Distribution (20/41)

Standard Normal Random Variable (Z): $\mu = 0 \quad \sigma^2 = 1$

The distribution of a normal random variable with **mean 0** and **variance 1** is called a ***standard normal distribution***.



Normal Distribution (21/41)

Density function of standard normal random variable (Z):

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \qquad -\infty < z < \infty$$

$$Z \sim N(0, 1)$$

μ : mean = 0

σ^2: variance = 1

Normal Distribution (22/41)

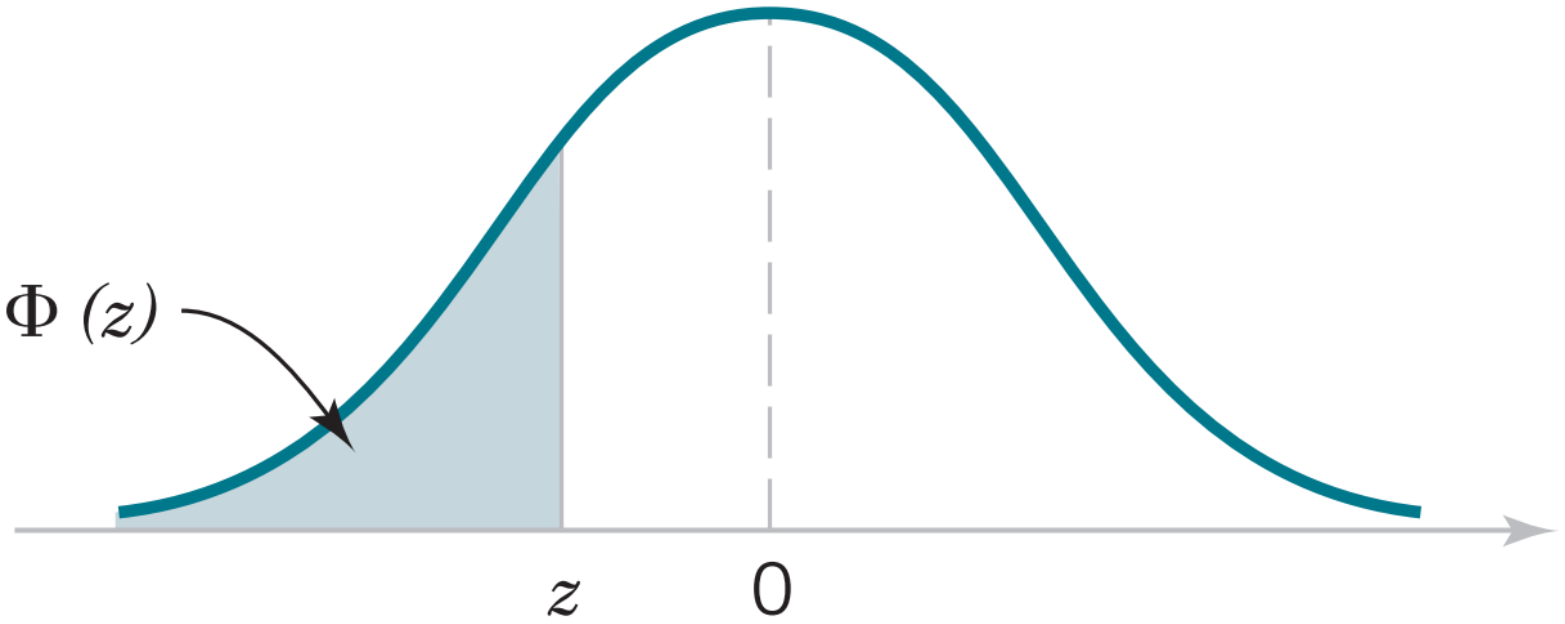
Cumulative distribution function of standard normal random variable (Z):

$$\Phi(z) = P(Z \leq z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-x^2/2} \, dv$$

$$Z \sim N(0, 1)$$

μ : mean = 0

σ^2: variance = 1



Normal Distribution (22/41)

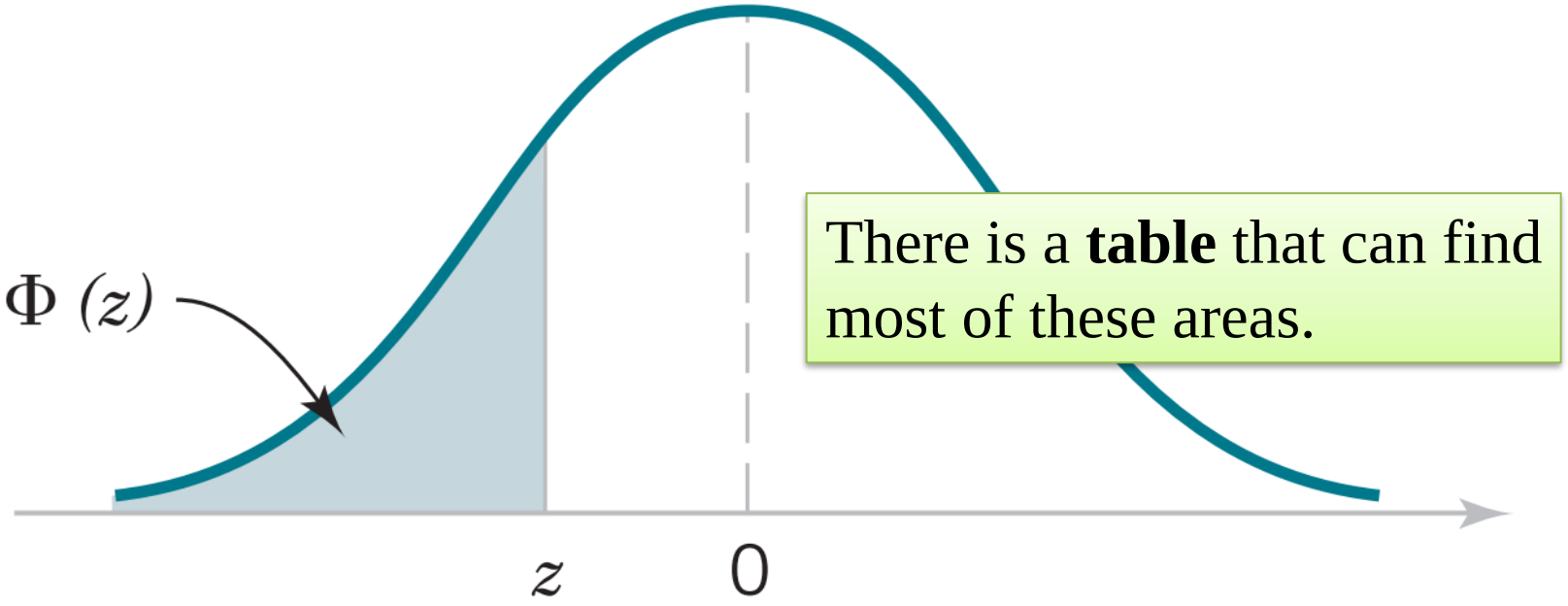
Cumulative distribution function of standard normal random variable (Z):

$$\Phi(z) = P(Z \leq z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-x^2/2} \, dv$$

$$Z \sim N(0, 1)$$

μ : mean = 0

σ^2: variance = 1



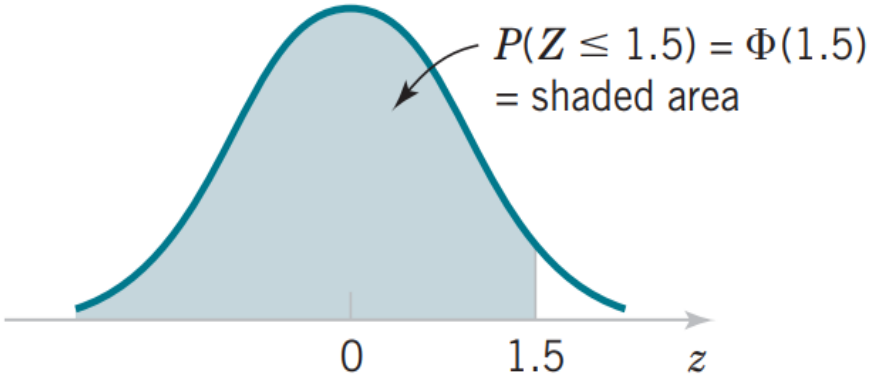
There is a **table** that can find most of these areas.

Normal Distribution (23/41)

Example2 (1/2):

Assume that Z is a standard normal random variable. The Table provides probabilities of the form $\Phi(z) = P(Z \leq z)$.

The use of Table to find $P(Z \leq 1.5)$ is illustrated as follows:



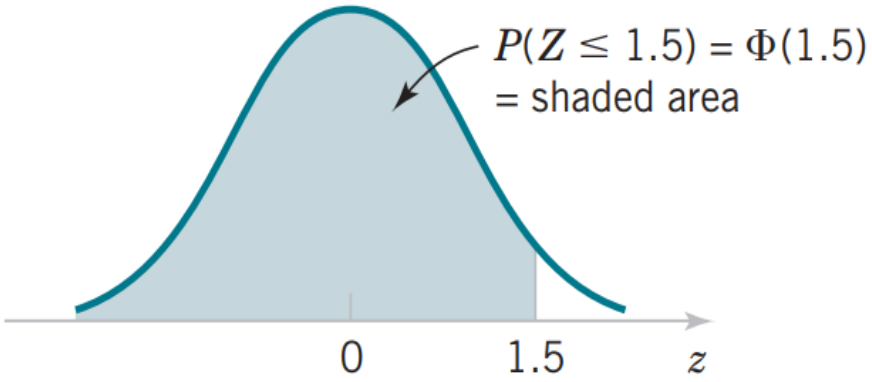
z	0.00	0.01	0.02	0.03
0	0.50000	0.50399	0.50800	0.51197
⋮		⋮		
1.5	0.93319	0.93448	0.93574	0.93699

Normal Distribution (23/41)

Example2 (2/2):

Assume that Z is a standard normal random variable. The Table provides probabilities of the form $\Phi(z) = P(Z \leq z)$.

The probability is 0.93319.



z	0.00	0.01	0.02	0.03
0	0.50000	0.50399	0.50800	0.51197
⋮	⋮	⋮	⋮	⋮
1.5	0.93319	0.93448	0.93574	0.93699

Normal Distribution (23/41)

Table A.3 Areas under the Normal Curve

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
−3.4	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0002
−3.3	0.0005	0.0005	0.0005	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0003
−3.2	0.0007	0.0007	0.0006	0.0006	0.0006	0.0006	0.0006	0.0005	0.0005	0.0005
−3.1	0.0010	0.0009	0.0009	0.0009	0.0008	0.0008	0.0008	0.0008	0.0007	0.0007
−3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
−2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
−2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
−2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
−2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
−2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
−2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
−2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
−2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
−2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
−2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
−1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
−1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
−1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
−1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
−1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559

Normal Distribution (23/41)

$\Phi(-2.35)$

Table A.3 Areas under the Normal Curve

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
−3.4	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0002
−3.3	0.0005	0.0005	0.0005	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0003
−3.2	0.0007	0.0007	0.0006	0.0006	0.0006	0.0006	0.0006	0.0005	0.0005	0.0005
−3.1	0.0010	0.0009	0.0009	0.0009	0.0008	0.0008	0.0008	0.0008	0.0007	0.0007
−3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
−2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
−2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
−2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
−2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
−2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
−2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
−2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
−2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
−2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
−2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
−1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
−1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
−1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
−1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
−1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559

Normal Distribution (23/41)

$$\Phi(-2.35) = P(Z \leq -2.35)$$

Table A.3 Areas under the Normal Curve

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
−3.4	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0002
−3.3	0.0005	0.0005	0.0005	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0003
−3.2	0.0007	0.0007	0.0006	0.0006	0.0006	0.0006	0.0006	0.0005	0.0005	0.0005
−3.1	0.0010	0.0009	0.0009	0.0009	0.0008	0.0008	0.0008	0.0008	0.0007	0.0007
−3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
−2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
−2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
−2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
−2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
−2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
−2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
−2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
−2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
−2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
−2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
−1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
−1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
−1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
−1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
−1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559

Normal Distribution (23/41)

$$\Phi(-2.35) = P(Z \leq -2.35)$$

Table A.3 Areas under the Normal Curve

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
−3.4	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0002
−3.3	0.0005	0.0005	0.0005	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0003
−3.2	0.0007	0.0007	0.0006	0.0006	0.0006	0.0006	0.0006	0.0005	0.0005	0.0005
−3.1	0.0010	0.0009	0.0009	0.0009	0.0008	0.0008	0.0008	0.0008	0.0007	0.0007
−3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
−2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
−2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
−2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
−2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
−2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
−2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
−2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
−2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
−2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
−2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
−1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
−1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
−1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
−1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
−1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559

Normal Distribution (23/41)

$$\Phi(-2.35) = P(Z \leq -2.35)$$

Table A.3 Areas under the Normal Curve

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
−3.4	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0002
−3.3	0.0005	0.0005	0.0005	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0003
−3.2	0.0007	0.0007	0.0006	0.0006	0.0006	0.0006	0.0006	0.0005	0.0005	0.0005
−3.1	0.0010	0.0009	0.0009	0.0009	0.0008	0.0008	0.0008	0.0008	0.0007	0.0007
−3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
−2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
−2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
−2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
−2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
−2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
−2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
−2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
−2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
−2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
−2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
−1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
−1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
−1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
−1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
−1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559

Normal Distribution (23/41)

$$\Phi(-2.35) = P(Z \leq -2.35) = 0.0094$$

Table A.3 Areas under the Normal Curve

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
−3.4	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0002
−3.3	0.0005	0.0005	0.0005	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0003
−3.2	0.0007	0.0007	0.0006	0.0006	0.0006	0.0006	0.0006	0.0005	0.0005	0.0005
−3.1	0.0010	0.0009	0.0009	0.0009	0.0008	0.0008	0.0008	0.0008	0.0007	0.0007
−3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
−2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
−2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
−2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
−2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
−2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
−2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
−2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
−2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
−2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
−2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
−1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
−1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
−1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
−1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
−1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559

Normal Distribution (23/41)

Table A.3 (continued) Areas under the Normal Curve

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767

Normal Distribution (23/41)

$$\Phi(1.27) = P(Z \leq 1.27)$$

Table A.3 (continued) Areas under the Norm

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767

Normal Distribution (23/41)

$$\Phi(1.27) = P(Z \leq 1.27) = 0.8980$$

Table A.3 (continued) Areas under the Norm

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767

Normal Distribution (23/41)

TABLE III Cumulative Standard Normal Distribution

z	−0.09	−0.08	−0.07	−0.06	−0.05	−0.04	−0.03	−0.02	−0.01	−0.00
−3.9	0.000033	0.000034	0.000036	0.000037	0.000039	0.000041	0.000042	0.000044	0.000046	0.000048
−3.8	0.000050	0.000052	0.000054	0.000057	0.000059	0.000062	0.000064	0.000067	0.000069	0.000072
−3.7	0.000075	0.000078	0.000082	0.000085	0.000088	0.000092	0.000096	0.000100	0.000104	0.000108
−3.6	0.000112	0.000117	0.000121	0.000126	0.000131	0.000136	0.000142	0.000147	0.000153	0.000159
−3.5	0.000165	0.000172	0.000179	0.000185	0.000193	0.000200	0.000208	0.000216	0.000224	0.000233
−3.4	0.000242	0.000251	0.000260	0.000270	0.000280	0.000291	0.000302	0.000313	0.000325	0.000337
−3.3	0.000350	0.000362	0.000376	0.000390	0.000404	0.000419	0.000434	0.000450	0.000467	0.000483
−3.2	0.000501	0.000519	0.000538	0.000557	0.000577	0.000598	0.000619	0.000641	0.000664	0.000687
−3.1	0.000711	0.000736	0.000762	0.000789	0.000816	0.000845	0.000874	0.000904	0.000935	0.000968
−3.0	0.001001	0.001035	0.001070	0.001107	0.001144	0.001183	0.001223	0.001264	0.001306	0.001350
−2.9	0.001395	0.001441	0.001489	0.001538	0.001589	0.001641	0.001695	0.001750	0.001807	0.001866
−2.8	0.001926	0.001988	0.002052	0.002118	0.002186	0.002256	0.002327	0.002401	0.002477	0.002555
−2.7	0.002635	0.002718	0.002803	0.002890	0.002980	0.003072	0.003167	0.003264	0.003364	0.003467
−2.6	0.003573	0.003681	0.003793	0.003907	0.004025	0.004145	0.004269	0.004396	0.004527	0.004661
−2.5	0.004799	0.004940	0.005085	0.005234	0.005386	0.005543	0.005703	0.005868	0.006037	0.006210
−2.4	0.006387	0.006569	0.006756	0.006947	0.007143	0.007344	0.007549	0.007760	0.007976	0.008198
−2.3	0.008424	0.008656	0.008894	0.009137	0.009387	0.009642	0.009903	0.010170	0.010444	0.010724
−2.2	0.011011	0.011304	0.011604	0.011911	0.012224	0.012545	0.012874	0.013209	0.013553	0.013903
−2.1	0.014262	0.014629	0.015003	0.015386	0.015778	0.016177	0.016586	0.017003	0.017429	0.017864
−2.0	0.018309	0.018763	0.019226	0.019699	0.020182	0.020675	0.021178	0.021692	0.022216	0.022750
−1.9	0.023295	0.023852	0.024419	0.024998	0.025588	0.026190	0.026803	0.027429	0.028067	0.028717

Normal Distribution (24/41)

Some Properties:

$\Phi(a) = P(Z \leq a) \quad \rightarrow \quad$ From Table

$P(Z < -a) = 1 - P(Z \leq a)$

$P(a < Z < b) = P(Z < b) - P(Z < a)$

$P(Z > a) = 1 - P(Z \leq a)$

$P(Z > -a) = P(Z < a)$

Normal Distribution (25/41)

Example3 (1/16):

Assume that Z is a standard normal random variable. Find the following probabilities:

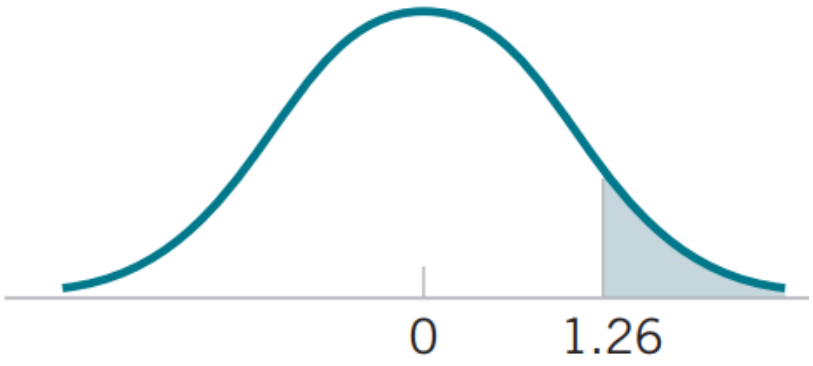
1. $P(Z > 1.26)$
2. $P(Z < -0.86)$
3. $P(Z > -1.37)$
4. $P(-1.25 < Z < 0.37)$
5. $P(Z \leq -4.6)$

Normal Distribution (25/41)

Example3 (2/16):

Assume that Z is a standard normal random variable. Find the following probabilities:

1. $P(Z > 1.26)$

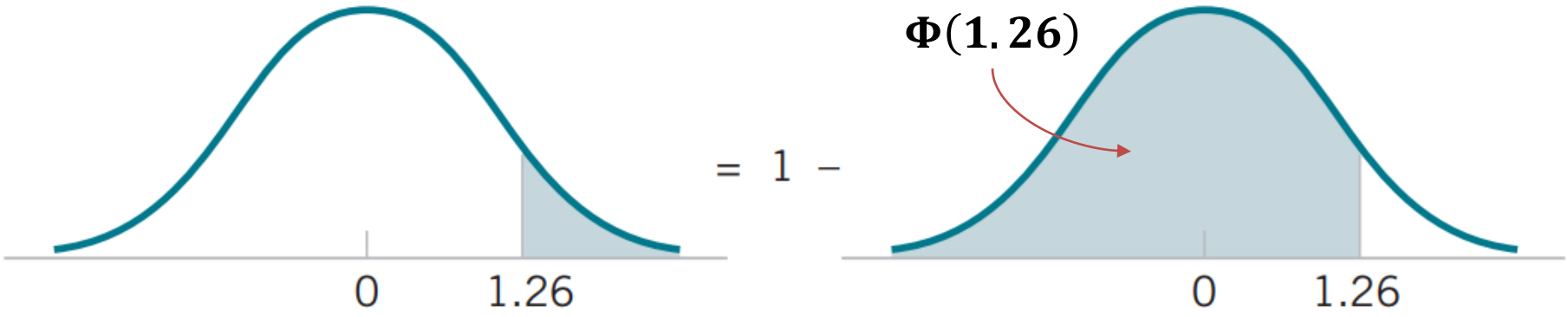


Normal Distribution (25/41)

Example3 (3/16):

Assume that Z is a standard normal random variable. Find the following probabilities:

1. $P(Z > 1.26) = 1 - \Phi(1.26)$



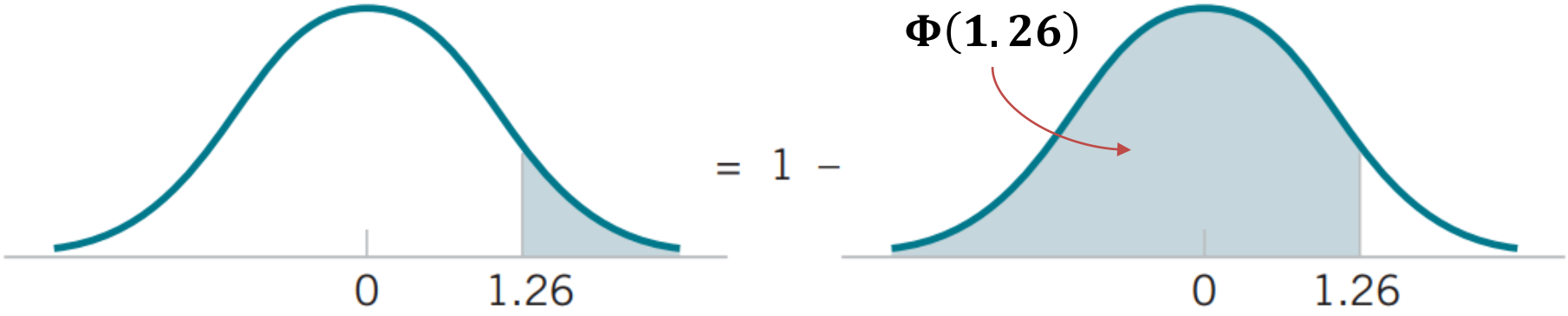
Normal Distribution (25/41)

Example3 (4/16):

Assume that Z is a st…ariable. Find the following probabilit…

$\Phi(1.26) = P(Z \leq 1.26)$
From the Table

1. $P(Z > 1.26) = 1 - \Phi(1.26)$



Normal Distribution (25/41)

Example3 (5/16):

$$\Phi(1.26) = 0.896165$$

TABLE III Cumulative Standard Normal Distribution (*continued*)

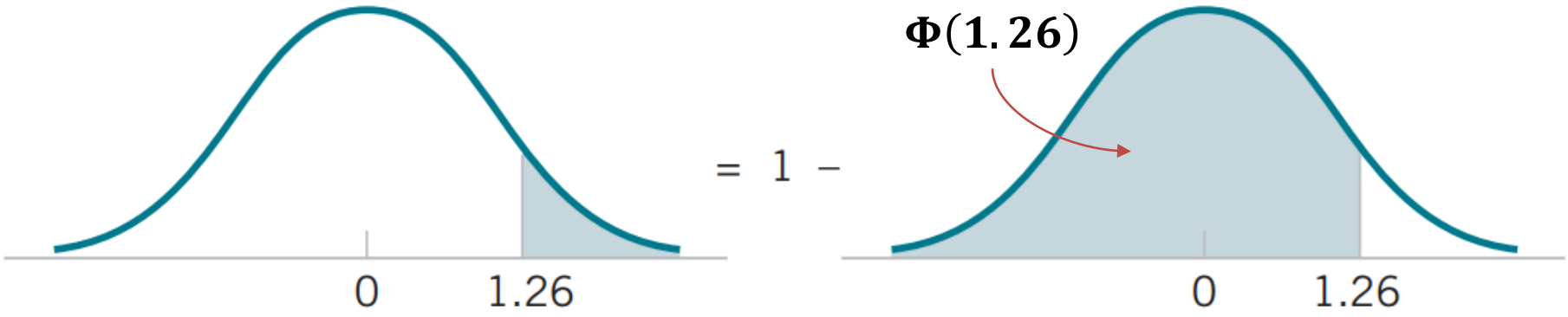
z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.500000	0.503989	0.507978	0.511967	0.515953	0.519939	0.532922	0.527903	0.531881	0.535856
0.1	0.539828	0.543795	0.547758	0.551717	0.555760	0.559618	0.563559	0.567495	0.571424	0.575345
0.2	0.579260	0.583166	0.587064	0.590954	0.594835	0.598706	0.602568	0.606420	0.610261	0.614092
0.3	0.617911	0.621719	0.625516	0.629300	0.633072	0.636831	0.640576	0.644309	0.648027	0.651732
0.4	0.655422	0.659097	0.662757	0.666402	0.670031	0.673645	0.677242	0.680822	0.684386	0.687933
0.5	0.691462	0.694974	0.698468	0.701944	0.705401	0.708840	0.712260	0.715661	0.719043	0.722405
0.6	0.725747	0.729069	0.732371	0.735653	0.738914	0.742154	0.745373	0.748571	0.751748	0.754903
0.7	0.758036	0.761148	0.764238	0.767305	0.770350	0.773373	0.776373	0.779350	0.782305	0.785236
0.8	0.788145	0.791030	0.793892	0.796731	0.799546	0.802338	0.805106	0.807850	0.810570	0.813267
0.9	0.815940	0.818589	0.821214	0.823815	0.826391	0.828944	0.831472	0.833977	0.836457	0.838913
1.0	0.841345	0.843752	0.846136	0.848495	0.850830	0.853141	0.855428	0.857690	0.859929	0.862143
1.1	0.864334	0.866500	0.868643	0.870762	0.872857	0.874928	0.876976	0.878999	0.881000	0.882977
1.2	0.884930	0.886860	0.888767	0.890651	0.892512	0.894350	0.896165	0.897958	0.899727	0.901475
1.3	0.903199	0.904902	0.906582	0.908241	0.909877	0.911492	0.913085	0.914657	0.916207	0.917736
1.4	0.919243	0.920730	0.922196	0.923641	0.925066	0.926471	0.927855	0.929219	0.930563	0.931888
1.5	0.933193	0.934478	0.935744	0.936992	0.938220	0.939429	0.940620	0.941792	0.942947	0.944083

Normal Distribution (25/41)

Example3 (6/16):

Assume that Z is a standard normal random variable. Find the following probabilities:

1. $P(Z > 1.26) = 1 - 0.896165 = 0.103835$

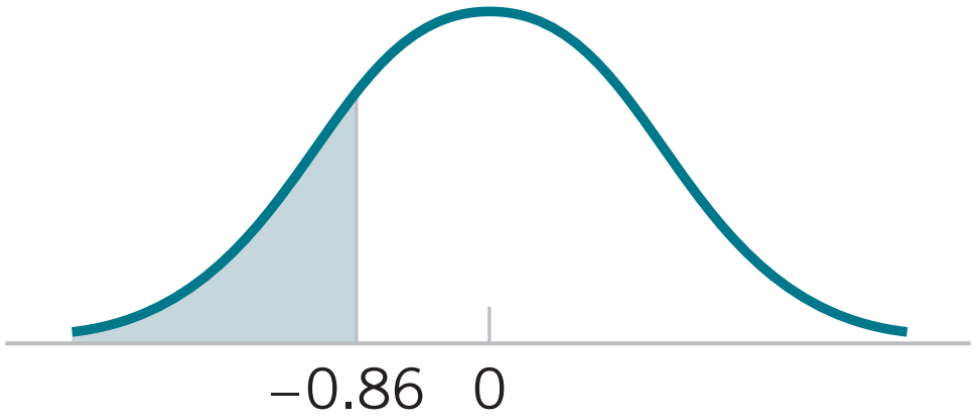


Normal Distribution (25/41)

Example3 (7/16):

Assume that Z is a standard normal random variable. Find the following probabilities:

2. $P(Z < -0.86)$

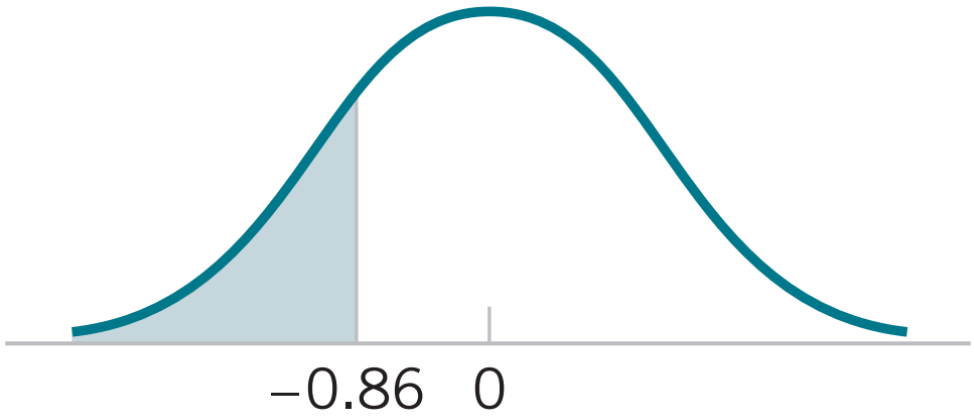


Normal Distribution (25/41)

Example3 (8/16):

Assume that Z is a standard normal random variable. Find the following probabilities:

2. $P(Z < -0.86) = \Phi(-0.86) = 0.19490$



Normal Distribution (25/41)

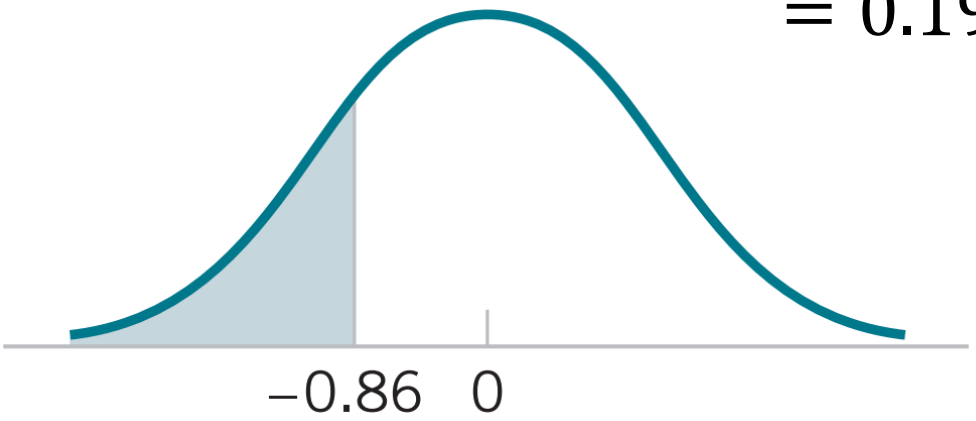
Example3 (9/16):

Assume that Z is a standard normal random variable. Find the following probabilities:

2. $P(Z < -0.86) = \Phi(-0.86) = 0.19490$

or $P(Z < -0.86) = 1 - P(Z < 0.86) = 1 - \Phi(0.86)$

$= 0.19490$

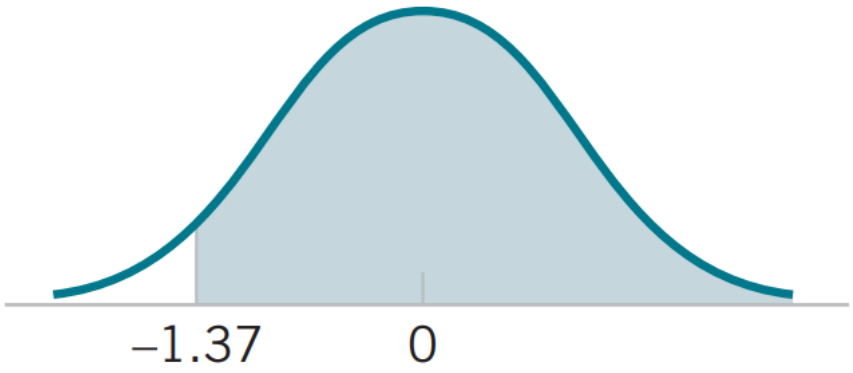


Normal Distribution (25/41)

Example3 (10/16):

Assume that Z is a standard normal random variable. Find the following probabilities:

3. $P(Z > -1.37)$

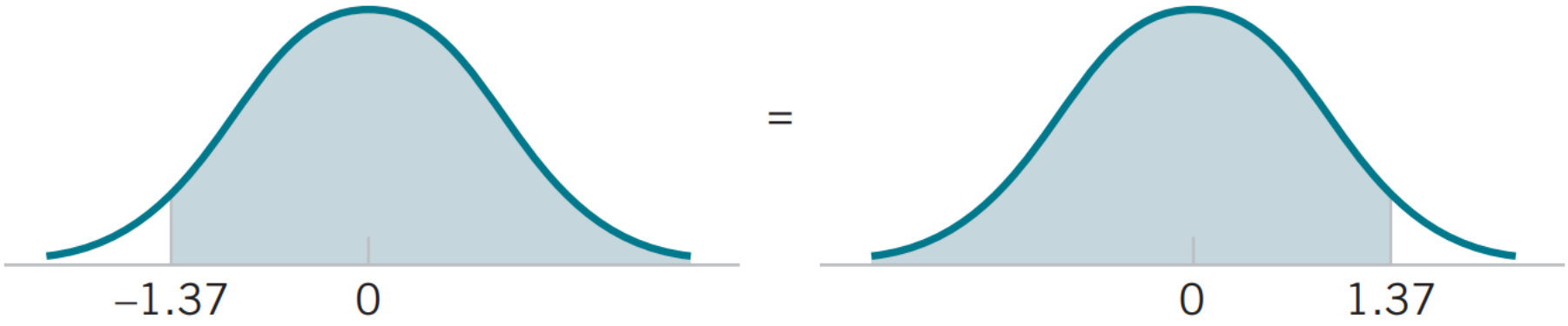


Normal Distribution (25/41)

Example3 (11/16):

Assume that Z is a standard normal random variable. Find the following probabilities:

3. $P(Z > -1.37) = P(Z < 1.37) = 0.91465$

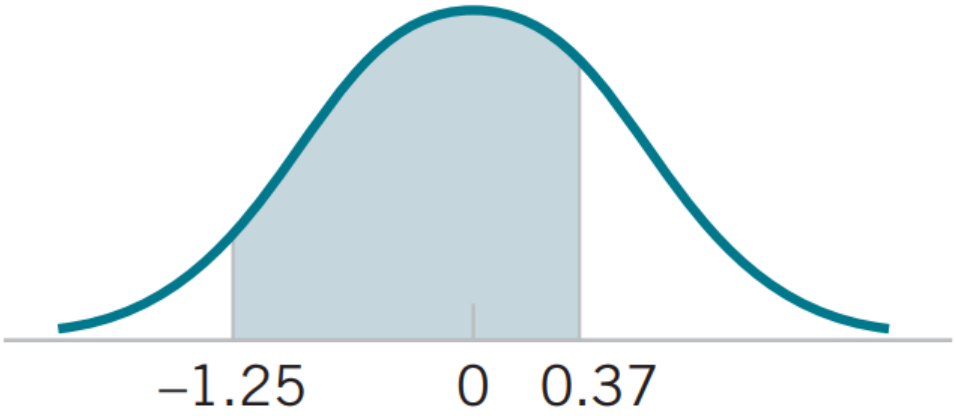


Normal Distribution (25/41)

Example3 (12/16):

Assume that Z is a standard normal random variable. Find the following probabilities:

4. $P(-1.25 < Z < 0.37)$

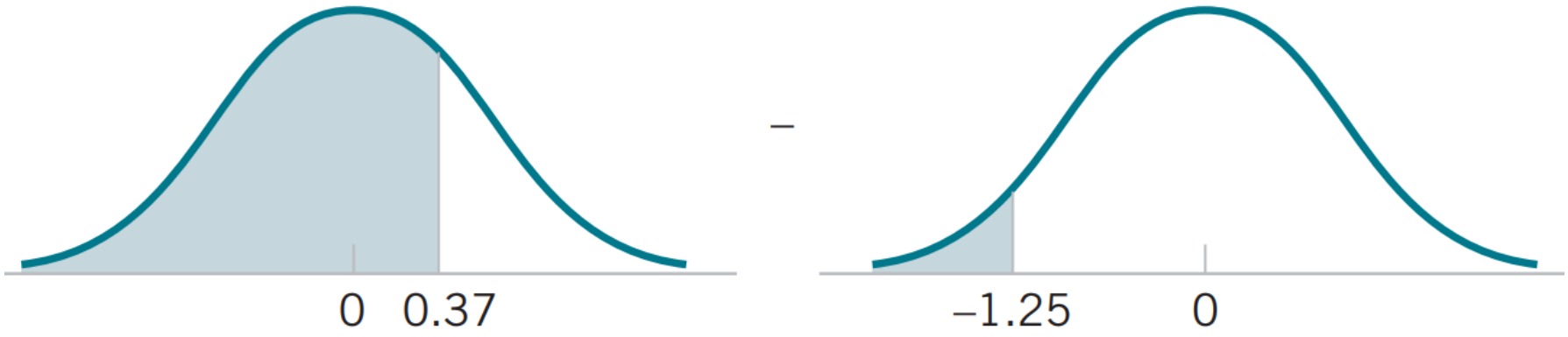


Normal Distribution (25/41)

Example3 (13/16):

Assume that Z is a standard normal random variable. Find the following probabilities:

4. $P(-1.25 < Z < 0.37) = \Phi(0.37) - \Phi(-1.25)$



Normal Distribution (25/41)

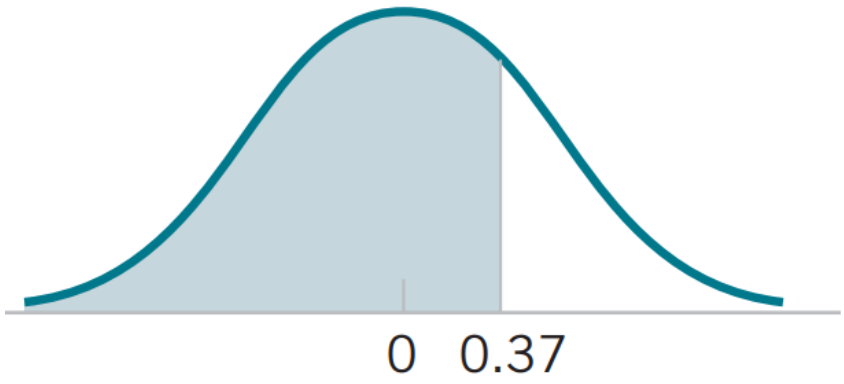
Example3 (14/16):

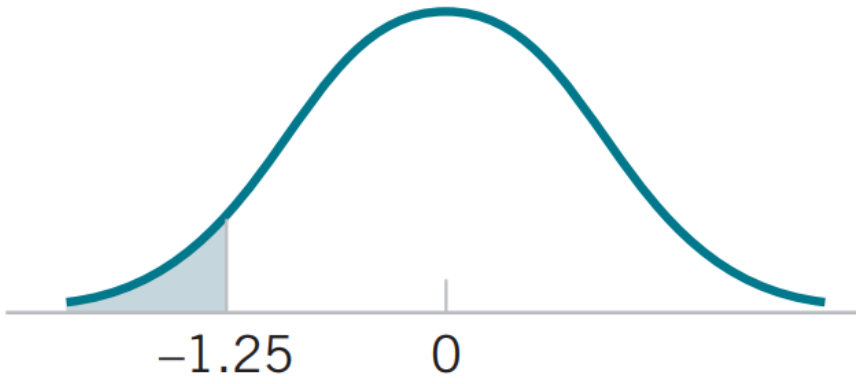
Assume that Z is a standard normal random variable. Find the following probabilities:

4. $P(-1.25 < Z < 0.37) = \Phi(0.37) - \Phi(-1.25)$

$= 0.64431 - 0.10565$

$= 0.53866$





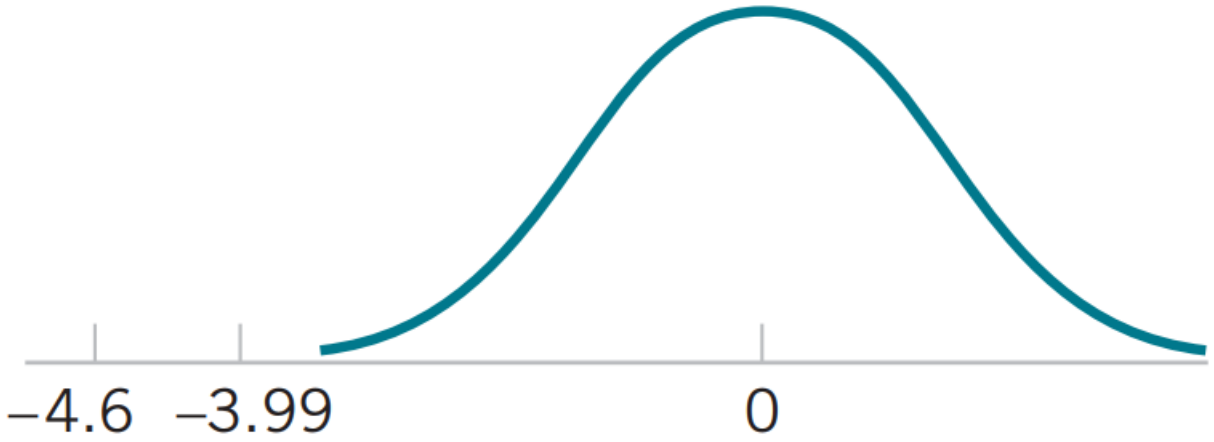
Normal Distribution (25/41)

Example3 (15/16):

Assume that Z is a standard normal random variable. Find the following probabilities:

5. $P(Z \leq -4.6)$ is nearly zero

The last entry in the table can be used to find that $P(Z \leq -3.99) = 0.00003$.



Normal Distribution (25/41)

Example3 (16/16):

Assume that Z is a standard normal random variable. Find the following probabilities:

5. $P(Z \leq -4.6)$ is nearly zero

The last entry in the table can be used to find that $P(Z \leq -3.99) = 0.00003$.

TABLE III Cumulative Standard Normal Distribution

z	−0.09	−0.08	−0.07	−0.06	−0.05	−0.04	−0.03	−0.02	−0.01	−0.00
−3.9	0.000033	0.000034	0.000036	0.000037	0.000039	0.000041	0.000042	0.000044	0.000046	0.000048
−3.8	0.000050	0.000052	0.000054	0.000057	0.000059	0.000062	0.000064	0.000067	0.000069	0.000072
−3.7	0.000075	0.000078	0.000082	0.000085	0.000088	0.000092	0.000096	0.000100	0.000104	0.000108

Normal Distribution (26/41)

Example4 (1/8):

Given a standard normal distribution Z, find the value of k such that

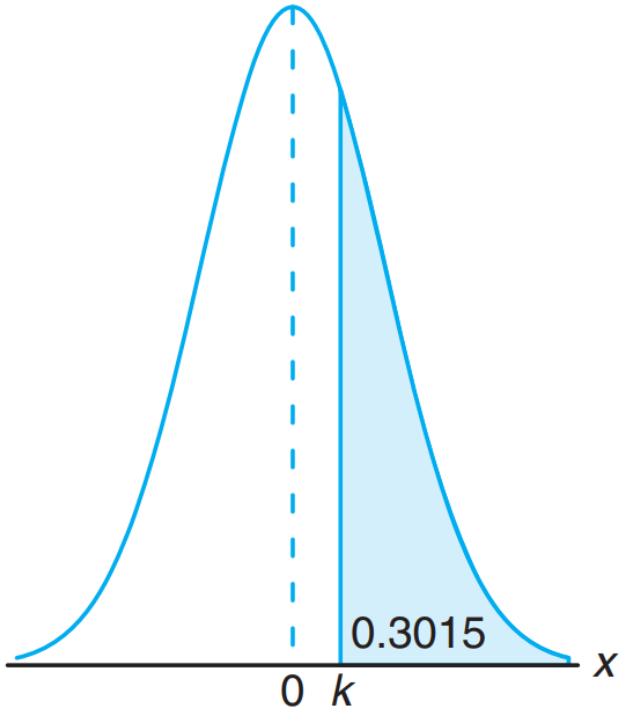
(a) $P(Z > k) = 0.3015$ and

(b) $P(k < Z < -0.18) = 0.4197$

Normal Distribution (26/41)

Example4 (2/8):

(a) $P(Z > k) = 0.3015$



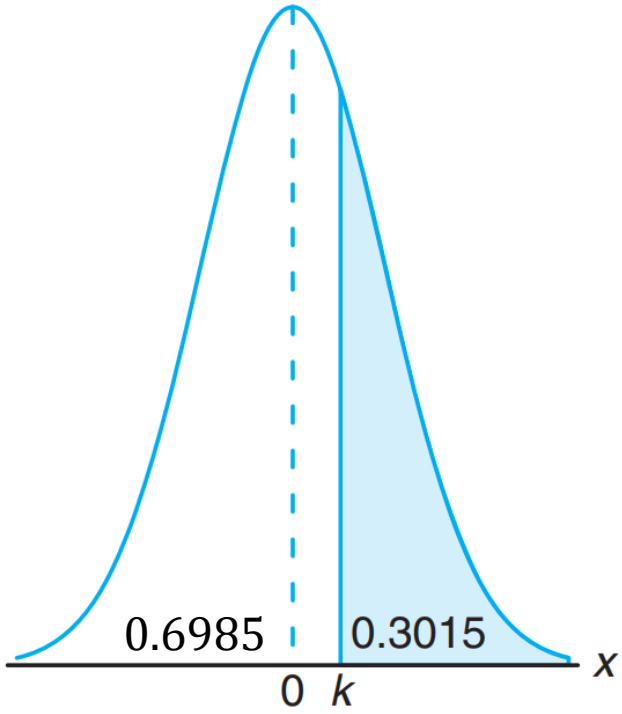
Normal Distribution (26/41)

Example4 (3/8):

(a) $P(Z > k) = 0.3015$

$P(Z < k) = 1 - 0.3015 = 0.6985$

From the Table



Normal Distribution (26/41)

Table A.3 (continued) Areas under the Normal Curve

z	.00	.01	.02	.03	.04	.05	.06
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962

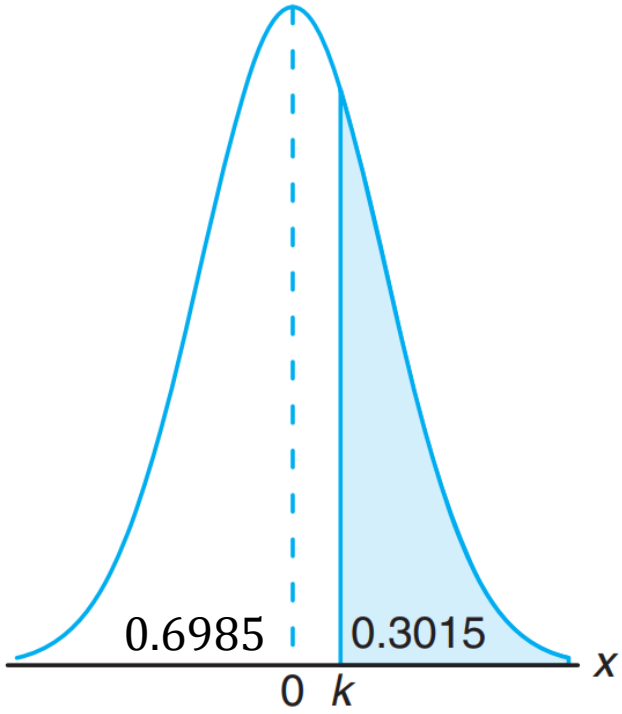
Normal Distribution (26/41)

Example4 (4/8):

(a) $P(Z > k) = 0.3015$

$P(Z < k) = 1 - 0.3015 = 0.6985$

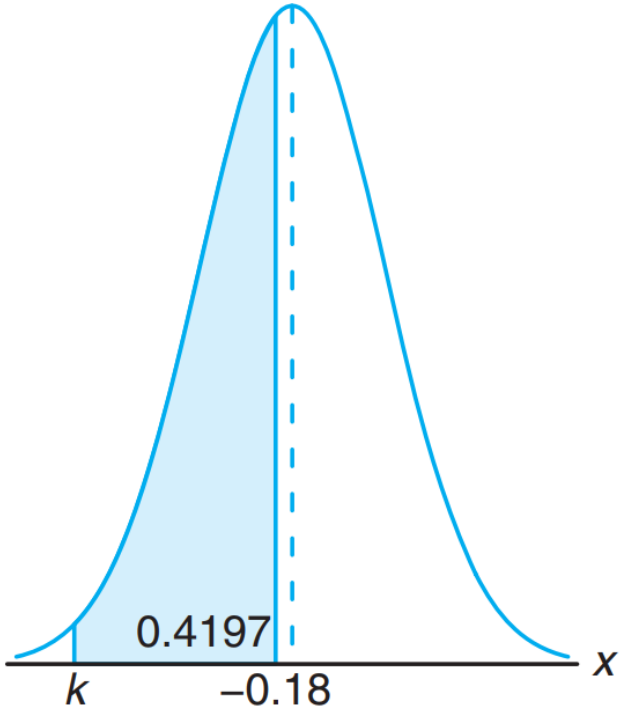
From the Table
it follows that $k = 0.52$.



Normal Distribution (26/41)

Example4 (5/8):

(b) $P(k < Z < -0.18) = 0.4197$



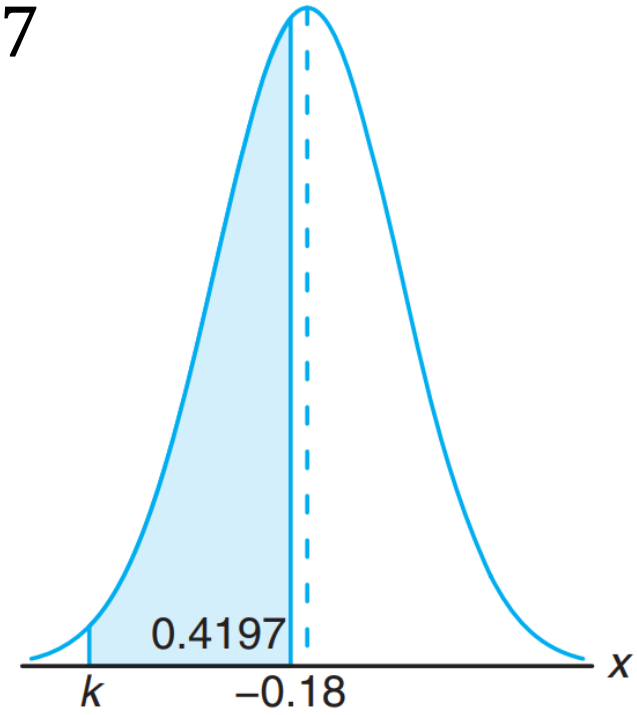
Normal Distribution (26/41)

Example4 (6/8):

(b) $P(k < Z < -0.18) = 0.4197$

$P(Z < -0.18) - P(Z < k) = 0.4197$

From the Table
$P(Z < -0.18) = 0.4286.$



Normal Distribution (26/41)

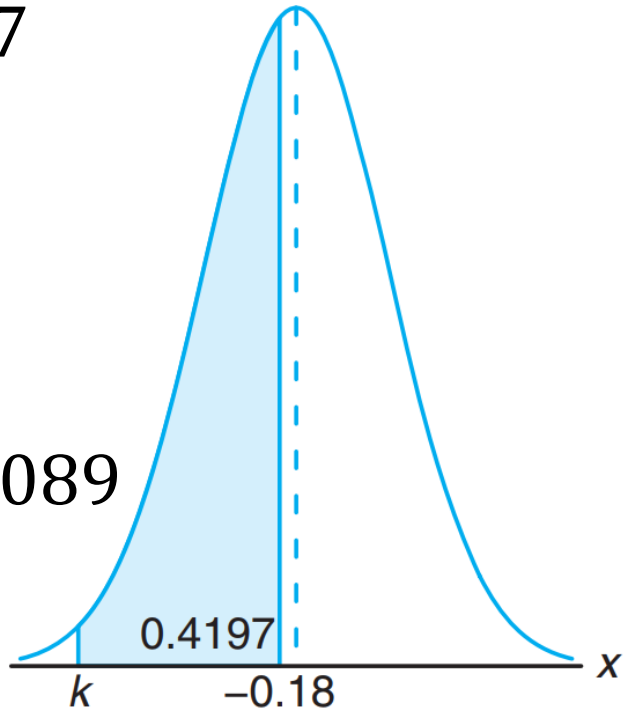
Example4 (7/8):

(b) $P(k < Z < -0.18) = 0.4197$

$P(Z < -0.18) - P(Z < k) = 0.4197$

From the Table
$P(Z < -0.18) = 0.4286.$

$\therefore 0.4286 - P(Z < k) = 0.4197$
$P(Z < k) = 0.4286 - 0.4197 = 0.0089$



Normal Distribution (26/41)

Table A.3 Areas under the Normal Curve

z	.00	.01	.02	.03	.04	.05	.06	.07	.08
−3.4	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003
−3.3	0.0005	0.0005	0.0005	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004
−3.2	0.0007	0.0007	0.0006	0.0006	0.0006	0.0006	0.0006	0.0005	0.0005
−3.1	0.0010	0.0009	0.0009	0.0009	0.0008	0.0008	0.0008	0.0008	0.0007
−3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010
−2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014
−2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020
−2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027
−2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037
−2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049
−2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066
−2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087
−2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113
−2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146

Normal Distribution (26/41)

Example4 (8/8):

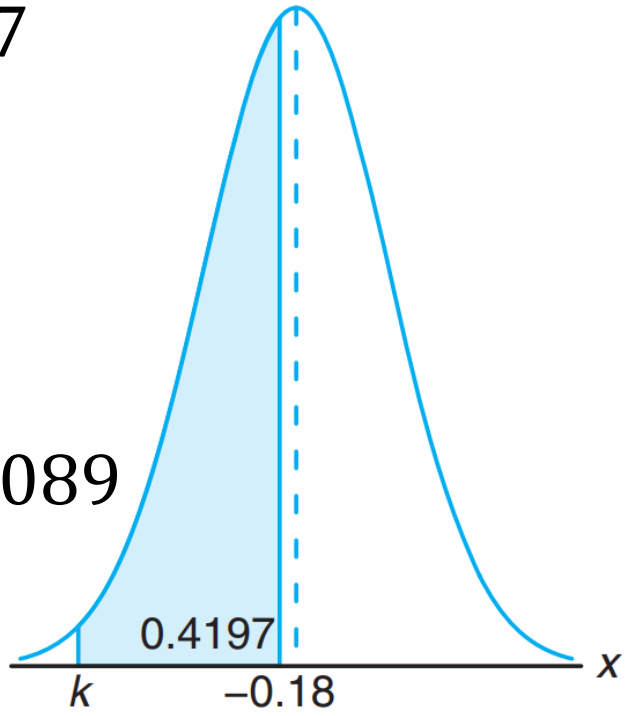
(b) $P(k < Z < -0.18) = 0.4197$

$P(Z < -0.18) - P(Z < k) = 0.4197$

From the Table
$P(Z < -0.18) = 0.4286.$

$\therefore 0.4286 - P(Z < k) = 0.4197$
$P(Z < k) = 0.4286 - 0.4197 = 0.0089$

From the Table
it follows that $k = -2.37.$



Normal Distribution (27/41)

Example5 (1/6):

Given a standard normal distribution Z, find the value of k such that

(a) $P(Z > k) = 0.05$ and

(b) $P(-k < Z < k) = 0.99$.

Normal Distribution (27/41)

Example5 (2/6):

(a) $P(Z > k) = 0.05$

$P(Z > k) = 1 - P(Z < k) = 0.05$

$P(Z < k) = 0.95$

Normal Distribution (27/41)

TABLE III Cumulative Standard Normal Distribution (*continued*)

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.500000	0.503989	0.507978	0.511967	0.515953	0.519939	0.532922	0.527903	0.531881	0.535856
0.1	0.539828	0.543795	0.547758	0.551717	0.555760	0.559618	0.563559	0.567495	0.571424	0.575345
0.2	0.579260	0.583166	0.587064	0.590954	0.594835	0.598706	0.602568	0.606420	0.610261	0.614092
0.3	0.617911	0.621719	0.625516	0.629300	0.633072	0.636831	0.640576	0.644309	0.648027	0.651732
0.4	0.655422	0.659097	0.662757	0.666402	0.670031	0.673645	0.677242	0.680822	0.684386	0.687933
0.5	0.691462	0.694974	0.698468	0.701944	0.705401	0.708840	0.712260	0.715661	0.719043	0.722405
0.6	0.725747	0.729069	0.732371	0.735653	0.738914	0.742154	0.745373	0.748571	0.751748	0.754903
0.7	0.758036	0.761148	0.764238	0.767305	0.770350	0.773373	0.776373	0.779350	0.782305	0.785236
0.8	0.788145	0.791030	0.793892	0.796731	0.799546	0.802338	0.805106	0.807850	0.810570	0.813267
0.9	0.815940	0.818589	0.821214	0.823815	0.826391	0.828944	0.831472	0.833977	0.836457	0.838913
1.0	0.841345	0.843752	0.846136	0.848495	0.850830	0.853141	0.855428	0.857690	0.859929	0.862143
1.1	0.864334	0.866500	0.868643	0.870762	0.872857	0.874928	0.876976	0.878999	0.881000	0.882977
1.2	0.884930	0.886860	0.888767	0.890651	0.892512	0.894350	0.896165	0.897958	0.899727	0.901475
1.3	0.903199	0.904902	0.906582	0.908241	0.909877	0.911492	0.913085	0.914657	0.916207	0.917736
1.4	0.919243	0.920730	0.922196	0.923641	0.925066	0.926471	0.927855	0.929219	0.930563	0.931888
1.5	0.933193	0.934478	0.935744	0.936992	0.938220	0.939429	0.940620	0.941792	0.942947	0.944083
1.6	0.945201	0.946301	0.947384	0.948449	0.949497	0.950529	0.951543	0.952540	0.953521	0.954486
1.7	0.955435	0.956367	0.957284	0.958185	0.959071	0.959941	0.960796	0.961636	0.962462	0.963273

Normal Distribution (27/41)

TABLE III Cumulative Standard Normal Distribution (*continued*)

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.500000	0.503989	0.507978	0.511967	0.515953	0.519939	0.532922	0.527903	0.531881	0.535856
0.1	0.539828	0.543795	0.547758	0.551717	0.555760	0.559618	0.563559	0.567495	0.571424	0.575345
0.2	0.579260	0.583166	0.587064	0.590954	0.594835	0.598706	0.602568	0.606420	0.610261	0.614092
0.3	0.617911	0.621719	0.625516	0.629300	0.633072	0.636831	0.640576	0.644309	0.648027	0.651732
0.4	0.655422	0.659097	0.662757	0.666402	0.670031	0.673645	0.677242	0.680822	0.684386	0.687933
0.5	0.691462	0.694974	0.698468	0.701944	0.705401	0.708840	0.712260	0.715661	0.719043	0.722405
0.6	0.725747	0.729069	0.732371	0.735653	0.738914	0.742154	0.745373	0.748571	0.751748	0.754903
0.7	0.758036	0.761148	0.764238	0.767305	0.770350	0.773373	0.776373	0.779350	0.782305	0.785236
0.8	0.788145	0.791030	0.793892	0.796731	0.799546	0.802338	0.805106	0.807850	0.810570	0.813267
0.9	0.815940	0.818589	0.821214	0.823815	0.826391	0.828944	0.831472	0.833977	0.836457	0.838913
1.0	0.841345	0.843752	0.846136	0.848495	0.850830	0.853141	0.855428	0.857690	0.859929	0.862143
1.1	0.864334	0.866500	0.868643	0.870762	0.872857	0.874928	0.876976	0.878999	0.881000	0.882977
1.2	0.884930	0.886860	0.888767	0.890651	0.892512	0.894350	0.896165	0.897958	0.899727	0.901475
1.3	0.903199	0.904902	0.906582	0.908241	0.909877	0.911492	0.913085	0.914657	0.916207	0.917736
1.4	0.919243	0.920730	0.922196	0.923641	0.925066	0.926471	0.927855	0.929219	0.930563	0.931888
1.5	0.933193	0.934478	0.935744	0.936992	0.938220	0.939429	0.940620	0.941792	0.942947	0.944083
1.6	0.945201	0.946301	0.947384	0.948449	0.949497	0.950529	0.951543	0.952540	0.953521	0.954486
1.7	0.955435	0.956367	0.957284	0.958185	0.959071	0.959941	0.960796	0.961636	0.962462	0.963273

Normal Distribution (27/41)

Example5 (3/6):

(a) $P(Z > k) = 0.05$

$P(Z > k) = 1 - P(Z < k) = 0.05$
$P(Z < k) = 0.95$

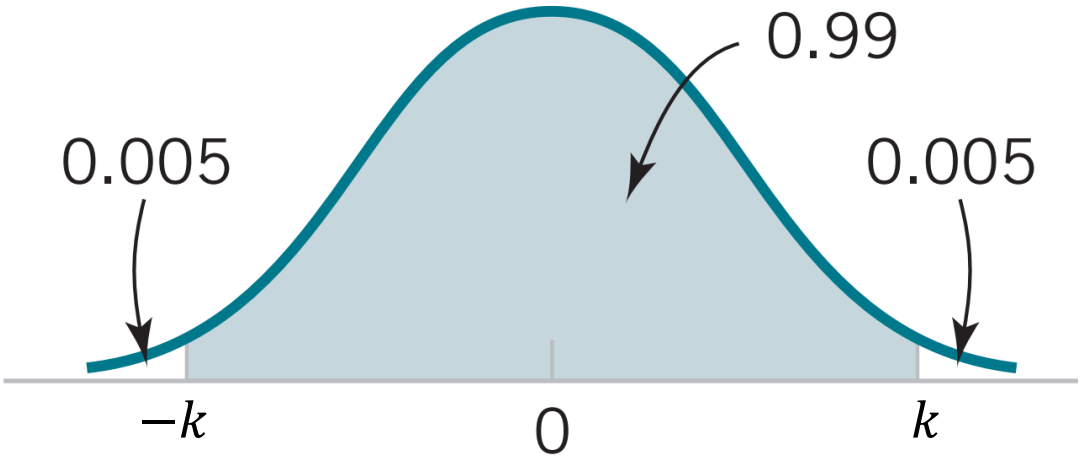
From the Table
We do not find 0.95 exactly; the nearest value is 0.95053, corresponding to $k \cong 1.65$.

Normal Distribution (27/41)

Example5 (4/6):

(b) $P(-k < Z < k) = 0.99.$

$$P(Z < k) - P(Z < -k) = 0.99$$



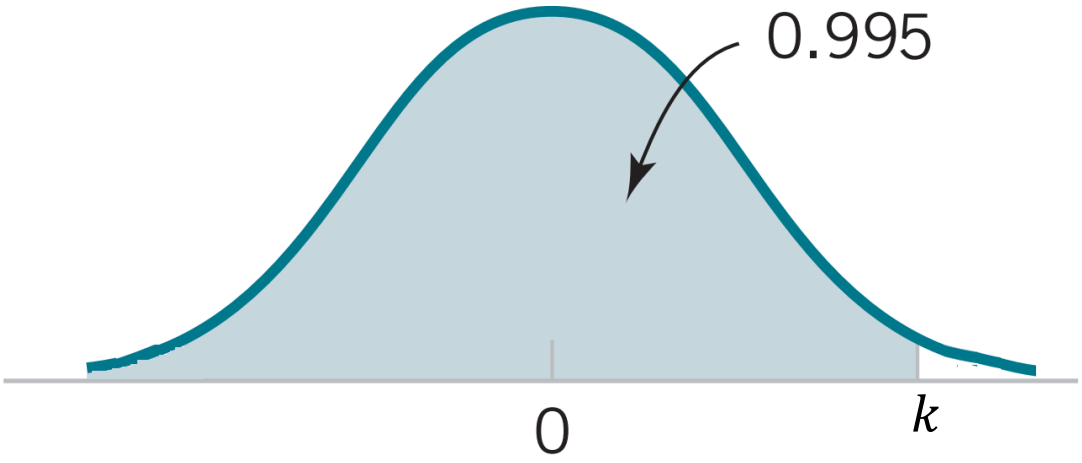
Normal Distribution (27/41)

Example5 (5/6):

(b) $P(-k < Z < k) = 0.99.$

$$P(Z < k) - P(Z < -k) = 0.99$$

$$P(Z < k) = 0.995$$



Normal Distribution (27/41)

TABLE III Cumulative Standard Normal Distribution (*continued*)

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.500000	0.503989	0.507978	0.511967	0.515953	0.519939	0.532922	0.527903	0.531881	0.535856
0.1	0.539828	0.543795	0.547758	0.551717	0.555760	0.559618	0.563559	0.567495	0.571424	0.575345
0.2	0.579260	0.583166	0.587064	0.590954	0.594835	0.598706	0.602568	0.606420	0.610261	0.614092
0.3	0.617911	0.621719	0.625516	0.629300	0.633072	0.636831	0.640576	0.644309	0.648027	0.651732
0.4	0.655422	0.659097	0.662757	0.666402	0.670031	0.673645	0.677242	0.680822	0.684386	0.687933
0.5	0.691462	0.694974	0.698468	0.701944	0.705401	0.708840	0.712260	0.715661	0.719043	0.722405
0.6	0.725747	0.729069	0.732371	0.735653	0.738914	0.742154	0.745373	0.748571	0.751748	0.754903
2.4	0.991802	0.992024	0.992240	0.992451	0.992656	0.992857	0.993053	0.993244	0.993431	0.993613
2.5	0.993790	0.993963	0.994132	0.994297	0.994457	0.994614	0.994766	0.994915	0.995060	0.995201
2.6	0.995339	0.995473	0.995604	0.995731	0.995855	0.995975	0.996093	0.996207	0.996319	0.996427
2.7	0.996533	0.996636	0.996736	0.996833	0.996928	0.997020	0.997110	0.997197	0.997282	0.997365

Normal Distribution (27/41)

TABLE III Cumulative Standard Normal Distribution (*continued*)

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.500000	0.503989	0.507978	0.511967	0.515953	0.519939	0.532922	0.527903	0.531881	0.535856
0.1	0.539828	0.543795	0.547758	0.551717	0.555760	0.559618	0.563559	0.567495	0.571424	0.575345
0.2	0.579260	0.583166	0.587064	0.590954	0.594835	0.598706	0.602568	0.606420	0.610261	0.614092
0.3	0.617911	0.621719	0.625516	0.629300	0.633072	0.636831	0.640576	0.644309	0.648027	0.651732
0.4	0.655422	0.659097	0.662757	0.666402	0.670031	0.673645	0.677242	0.680822	0.684386	0.687933
0.5	0.691462	0.694974	0.698468	0.701944	0.705401	0.708840	0.712260	0.715661	0.719043	0.722405
0.6	0.725747	0.729069	0.732371	0.735653	0.738914	0.742154	0.745373	0.748571	0.751748	0.754903
2.4	0.991802	0.992024	0.992240	0.992451	0.992656	0.992857	0.993053	0.993244	0.993431	0.993613
2.5	0.993790	0.993963	0.994132	0.994297	0.994457	0.994614	0.994766	0.994915	0.995060	0.995201
2.6	0.995339	0.995473	0.995604	0.995731	0.995855	0.995975	0.996093	0.996207	0.996319	0.996427
2.7	0.996533	0.996636	0.996736	0.996833	0.996928	0.997020	0.997110	0.997197	0.997282	0.997365

Normal Distribution (27/41)

Example5 (6/6):

(b) $P(-k < Z < k) = 0.99.$

$$P(Z < k) - P(Z < -k) = 0.99$$

$$P(Z < k) = 0.995$$

From the Table
We do not find 0.995 exactly; the nearest value is 0.99506, corresponding to $k \cong 2.58$.

Normal Distribution (28/41)

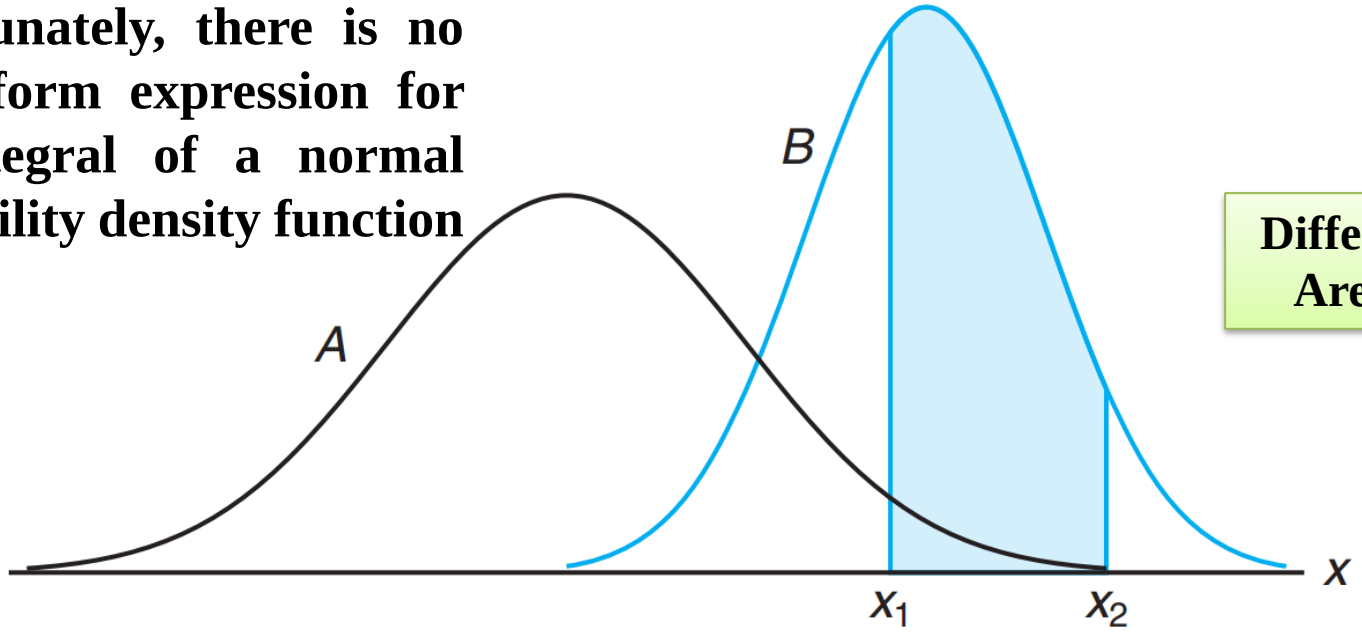
Recall

Areas under the Normal Curve:

Different means and standard deviations

$$P(x_1 < X < x_2) = \frac{1}{\sigma\sqrt{2\pi}} \int_{x_1}^{x_2} e^{\frac{-(x-\mu)^2}{2\sigma^2}} \, dx$$

Unfortunately, there is no closed-form expression for the integral of a normal probability density function



Different Areas

Normal Distribution (29/41)

Standardizing a Normal Random Variable:

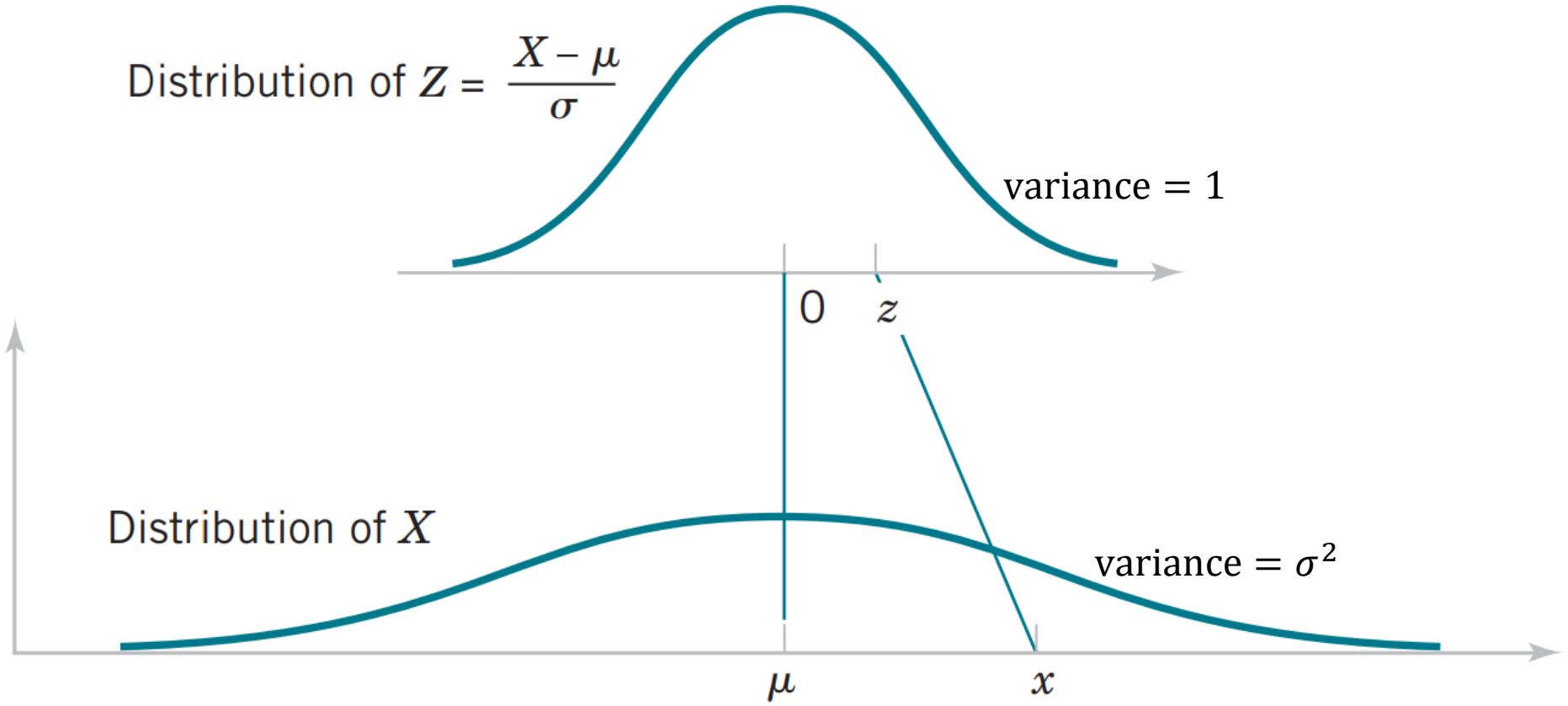
If X is a normal random variable with mean μ and variance σ^2, the random variable

$$Z = \frac{X - \mu}{\sigma}$$

is a normal random variable with mean = 0 and variance = 1. That is, Z is a standard normal random variable.

Normal Distribution (30/41)

$$X \sim N(\mu, \sigma^2) \implies Z = \frac{X-\mu}{\sigma} \sim N(0,1)$$



Normal Distribution (31/41)

The relationship between the values of X and the transformed values of Z:

$$X \sim N(\mu, \sigma^2) \Rightarrow Z = \frac{X-\mu}{\sigma} \sim N(0,1)$$

The value z is

$$z = \frac{x - \mu}{\sigma}$$

And the value x is

$$x = \sigma z + \mu$$

Normal Distribution (32/41)

Example1 (1/7):

Suppose that the current measurements in a strip of wire are assumed to follow a normal distribution with a mean of 10 milliamperes and a variance of 4 $(\text{milliamperes})^2$. What is the probability that a measurement exceeds 13 milliamperes?

Normal Distribution (32/41)

Example1 (2/7):

Suppose that the current measurements in a strip of wire are assumed to follow a normal distribution with a mean of 10 milliamperes and a variance of 4 (milliamperes)^2. What is the probability that a measurement exceeds 13 milliamperes?

Let X denote the current in milliamperes. The requested probability can be represented as $P(X > 13)$.

$\mu = 10$ and $\sigma^2 = 4 \quad \rightarrow \quad \sigma = 2$

Normal Distribution (32/41)

Example1 (3/7):

$P(X > 13)$

$\mu = 10$ and $\sigma^2 = 4 \quad \rightarrow \quad \sigma = 2$

$$P(X > 13) = P\left(Z > \frac{13 - \mu}{\sigma}\right) = P\left(Z > \frac{13 - 10}{2}\right)$$

$$= P(Z > 1.5)$$

Normal Distribution (32/41)

Example1 (4/7):

$$P(X > 13) = P(Z > 1.5)$$

$$= 1 - P(Z \leq 1.5)$$

Normal Distribution (32/41)

Example1 (5/7):

$$P(X > 13) = P(Z > 1.5)$$

$$= 1 - P(Z \leq 1.5)$$

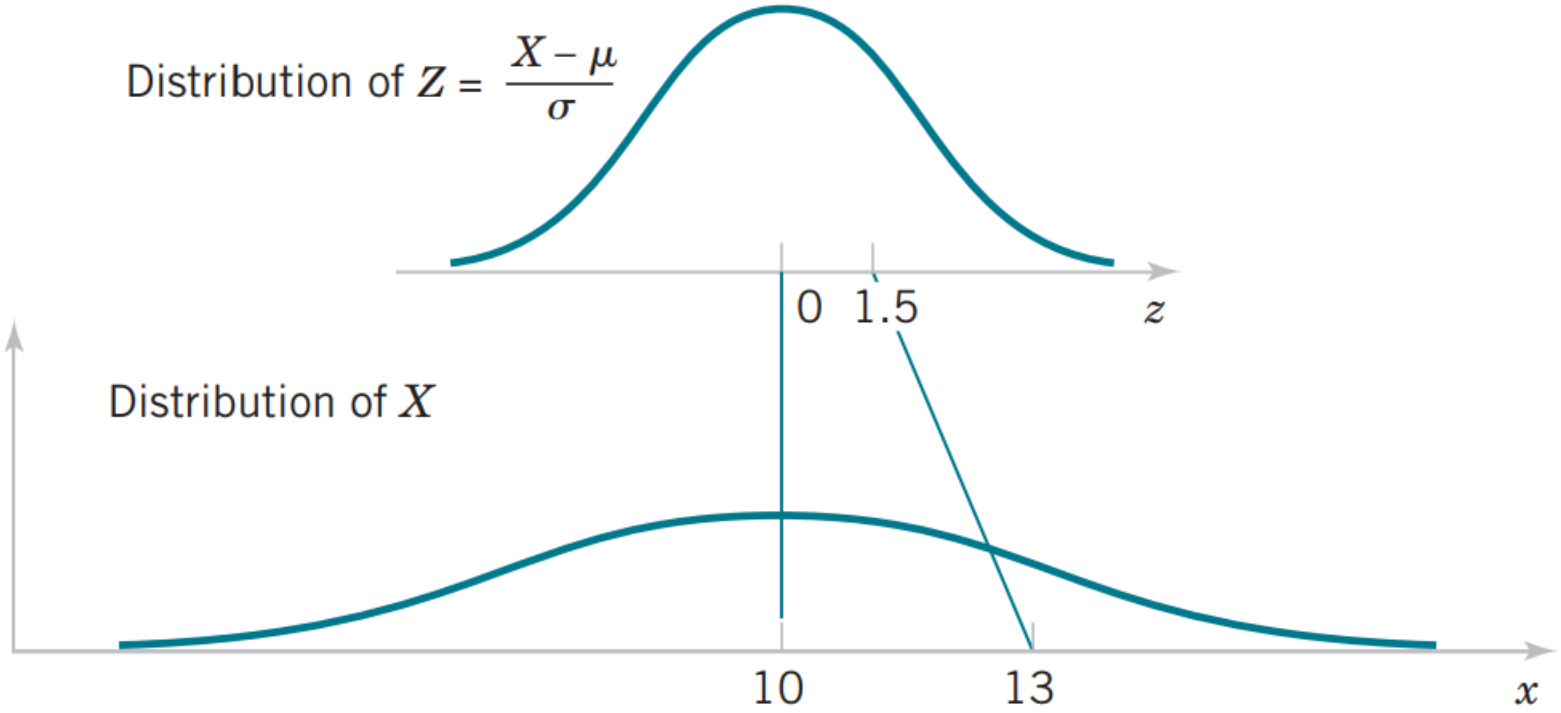
From the Table $P(Z \leq 1.5) = 0.93319$

$$\therefore P(X > 13) = 1 - 0.93319 = 0.06681$$

Normal Distribution (32/41)

Example1 (6/7):

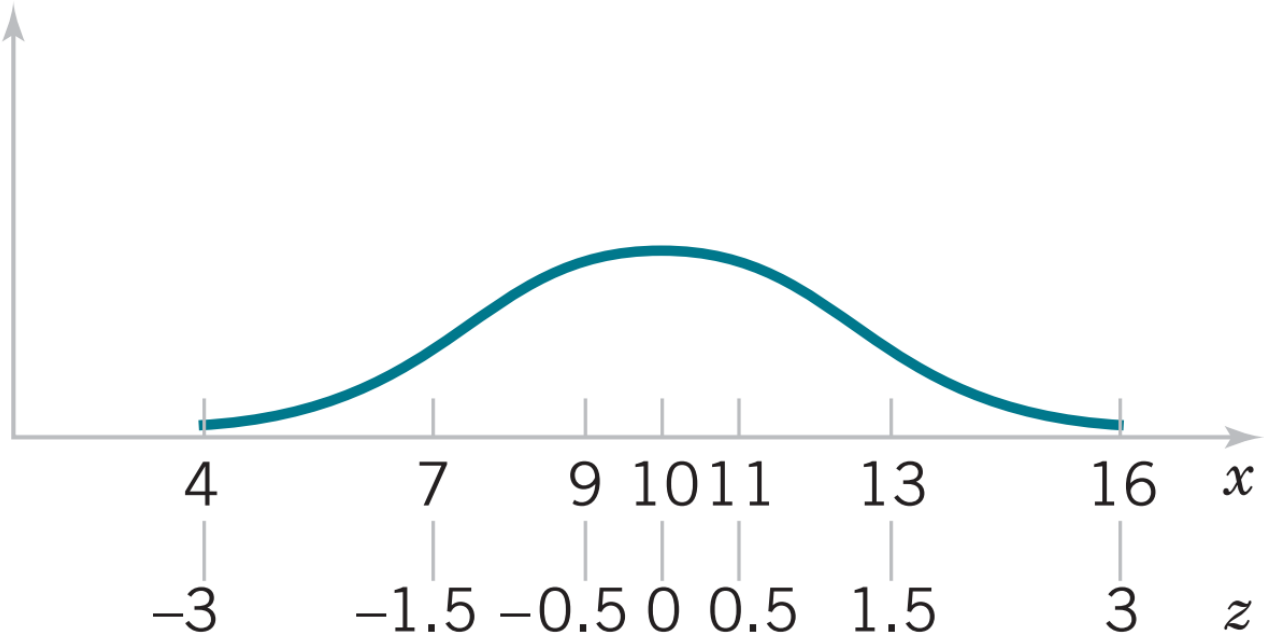
$$P(X > 13) = P(Z > 1.5) = 0.06681$$



Normal Distribution (32/41)

Example1 (7/7):

$$P(X > 13) = P(Z > 1.5) = 0.06681$$



Normal Distribution (33/41)

Example2 (1/6):

Continuing Example 1, what is the probability that a current measurement is between 9 and 11 milliamperes?

$$P\,(9 < X < 11)$$

Then, determine the value for which the probability that a current measurement is less than this value is 0.98.

$$P(X < x) = 0.98$$

Normal Distribution (33/41)

Example2 (2/6):

Continuing Example 1, what is the probability that a current measurement is between 9 and 11 milliamperes?

$$P(9 < X < 11) = P\left(\frac{9-10}{2} < \frac{X-10}{2} < \frac{11-10}{2}\right)$$

Normal Distribution (33/41)

Example2 (3/6):

Continuing Example 1, what is the probability that a current measurement is between 9 and 11 milliamperes?

$$P(9 < X < 11) = P\left(\frac{9-10}{2} < \frac{X-10}{2} < \frac{11-10}{2}\right)$$
$$= P(-0.5 < Z < 0.5)$$
$$= P(Z < 0.5) - P(Z < -0.5)$$
$$= 0.69146 - 0.30854 = 0.38292$$

Normal Distribution (33/41)

Example2 (4/6):

Determine the value for which the probability that a current measurement is less than this value is 0.98.

$$P(X < x) = 0.98$$

$$P(X < x) = P\left(\frac{X - 10}{2} < \frac{x - 10}{2}\right)$$

$$= P\left(Z < \frac{x - 10}{2}\right) = 0.98$$

Normal Distribution (33/41)

Example2 (5/6):

$$P(X < x) = P\left(\frac{X - 10}{2} < \frac{x - 10}{2}\right)$$
$$= P\left(Z < \frac{x - 10}{2}\right) = 0.98$$

Table is used to find the z-value such that $P(Z < z) = 0.98$. The nearest probability from the Table results in $P(Z < 2.06) = 0.980301$.

Normal Distribution (33/41)

Example2 (6/6):

$$P(X < x) = P\left(\frac{X-10}{2} < \frac{x-10}{2}\right)$$
$$= P\left(Z < \frac{x-10}{2}\right) = 0.98$$

Table is used to find the z-value such that $P(Z < z) = 0.98$. The nearest probability from the Table results in $P(Z < 2.06) = 0.980301$.

$\because x = \sigma z + \mu$

$\therefore x = (2)(2.06) + 10 = 14.1$ mA

Normal Distribution (34/41)

Example3 (1/8):

The loaves of rye bread distributed to local stores by a certain bakery have an average length of 30 centimeters and a standard deviation of 2 centimeters. Assuming that the lengths are normally distributed,

what percentage of the loaves are

(a) longer than 31.7 centimeters?
(b) between 29.3 and 33.5 centimeters in length?

Normal Distribution (34/41)

Example3 (2/8):

$$X \sim N(\mu, \sigma^2) \Rightarrow Z = \frac{X-\mu}{\sigma} \sim N(0,1)$$

The loaves of rye bread distributed to local stores by a certain bakery have an **average length of 30** centimeters and a **standard deviation of 2** centimeters. Assuming that the lengths are *normally distributed*, what percentage of the loaves are

(a) longer than 31.7 centimeters?
(b) between 29.3 and 33.5 centimeters in length?

Normal Distribution (34/41)

Example3 (3/8):

$$X \sim N(30, 2^2) \Rightarrow Z = \frac{X-30}{2} \sim N(0,1)$$

The loaves of rye bread distributed to local stores by a certain bakery have an **average length of 30** centimeters and a **standard deviation of 2** centimeters. Assuming that the lengths are *normally distributed*, what percentage of the loaves are

(a) longer than 31.7 centimeters?
(b) between 29.3 and 33.5 centimeters in length?

Normal Distribution (34/41)

Example3 (4/8):

$$X \sim N(30, 2^2) \Rightarrow Z = \frac{X-30}{2} \sim N(0,1)$$

(a) longer than 31.7 centimeters?

$$P(X > 31.7) = P\left(Z > \frac{31.7-\mu}{\sigma}\right) = P\left(Z > \frac{31.7-30}{2}\right)$$

$$P(Z > 0.85) = 1 - P(Z \leq 0.85)$$

Normal Distribution (34/41)

$$P(Z \leq 0.85) = 0.8023$$

Table A.3 (continued) Areas under the Normal Curve

z	.00	.01	.02	.03	.04	.05	.06
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315

Normal Distribution (34/41)

Example3 (5/8):

$$X \sim N(30, 2^2) \Rightarrow Z = \frac{X-30}{2} \sim N(0,1)$$

(a) longer than 31.7 centimeters?

$$P(X > 31.7) = P\left(Z > \frac{31.7-\mu}{\sigma}\right) = P\left(Z > \frac{31.7-30}{2}\right)$$

$$P(Z > 0.85) = 1 - P(Z \leq 0.85) = 1 - 0.8023 = 0.1977$$

Then, 19.77% of the loaves are longer than 31.7 centimeters

Normal Distribution (34/41)

Example3 (6/8):

$$X \sim N(30, 2^2) \Rightarrow Z = \frac{X-30}{2} \sim N(0,1)$$

(b) between 29.3 and 33.5 centimeters in length?

$P(29.3 < X < 33.5)$

Normal Distribution (34/41)

Example3 (7/8):

$$X \sim N(30, 2^2) \Rightarrow Z = \frac{X-30}{2} \sim N(0,1)$$

(b) between 29.3 and 33.5 centimeters in length?

$$P(29.3 < X < 33.5) = P\left(\frac{29.3-30}{2} < Z < \frac{33.5-30}{2}\right)$$

$$= P(-0.35 < Z < 1.75)$$

Normal Distribution (34/41)

Example3 (8/8):

$$X \sim N(30, 2^2) \Rightarrow Z = \frac{X-30}{2} \sim N(0,1)$$

(b) between 29.3 and 33.5 centimeters in length?

$$P(29.3 < X < 33.5) = P\left(\frac{29.3-30}{2} < Z < \frac{33.5-30}{2}\right)$$

$$= P(-0.35 < Z < 1.75)$$

$$P(-0.35 < Z < 1.75) = P(Z < 1.75) - P(Z < -0.35)$$

$$P(-0.35 < Z < 1.75) = 0.9599 - 0.3632 = 0.5967$$

Then, 59.67% of the loaves are between 29.3 and 33.5 centimeters.

Normal Distribution (35/41)

Example5 (1/8):

The average grade for an exam is 74, and the standard deviation is 7. If 12% of the class is given *A*s, and the grades are curved to follow a normal distribution, what is the lowest possible integer grade for *A* and the highest possible integer grade for *B*?

Normal Distribution (35/41)

Example5 (2/8):

$$X \sim N(74, 7^2) \Rightarrow Z = \frac{X-74}{7} \sim N(0,1)$$

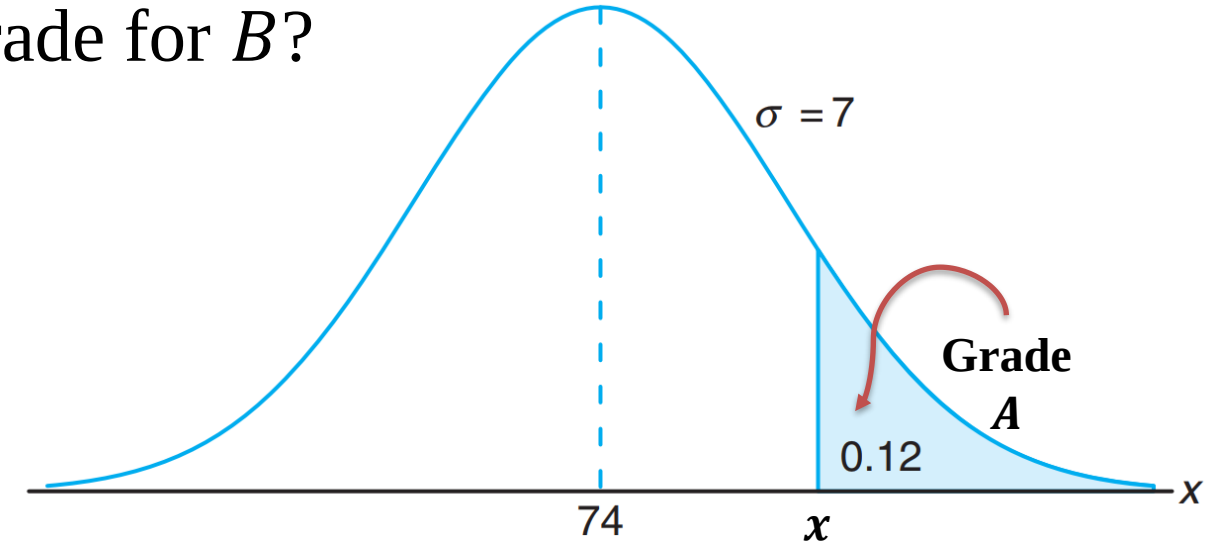
The **average grade for an exam is 74**, and the **standard deviation is 7**. If 12% of the class is given *A*s, and the grades are curved to follow a normal distribution, what is the lowest possible integer grade for *A* and the highest possible integer grade for *B*?

Normal Distribution (35/41)

Example5 (3/8):

$$X \sim N(74, 7^2) \Rightarrow Z = \frac{X-74}{7} \sim N(0,1)$$

The average grade for an exam is 74, and the standard deviation is 7. If **12% of the class is given *A*s**, and the grades are curved to follow a normal distribution, what is the lowest possible integer grade for *A* and the highest possible integer grade for *B*?

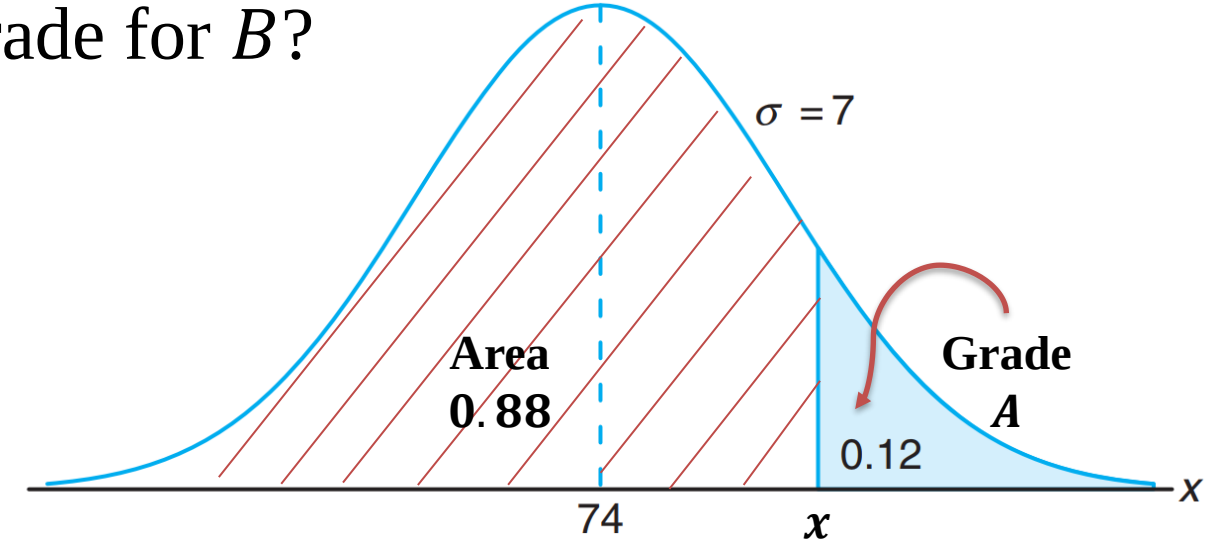


Normal Distribution (35/41)

Example5 (4/8):

$$X \sim N(74, 7^2) \Rightarrow Z = \frac{X-74}{7} \sim N(0,1)$$

The average grade for an exam is 74, and the standard deviation is 7. If **12% of the class is given *A*s**, and the grades are curved to follow a normal distribution, what is the lowest possible integer grade for *A* and the highest possible integer grade for *B*?



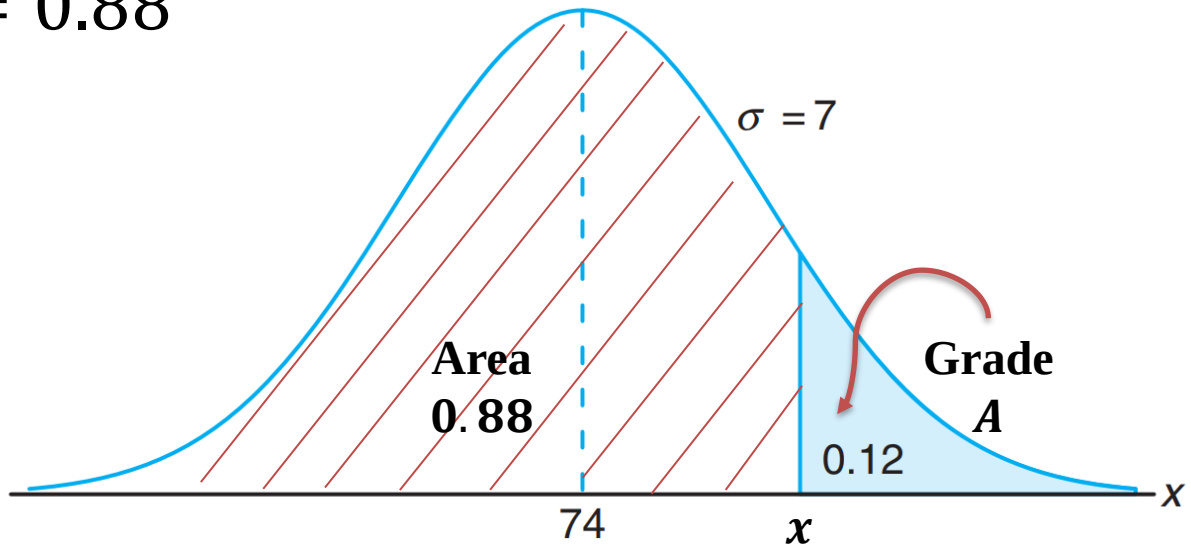
Normal Distribution (35/41)

Example5 (5/8):

$$X \sim N(74, 7^2) \Rightarrow Z = \frac{X-74}{7} \sim N(0,1)$$

$$P(X < x) = 0.88$$

$$P\left(Z < \frac{x-74}{7}\right) = 0.88$$



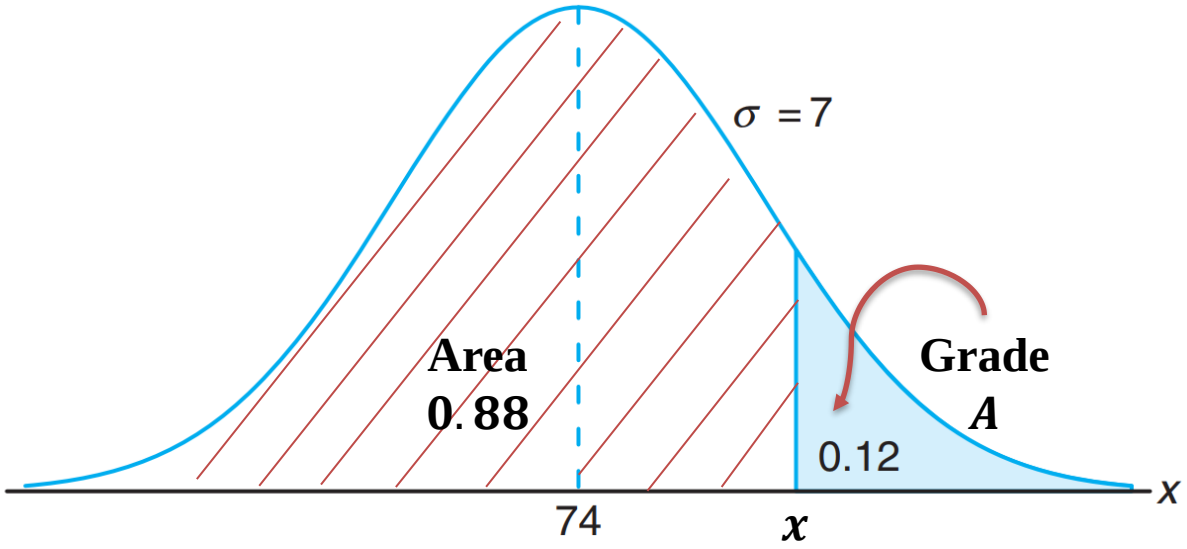
Normal Distribution (35/41)

Example5 (6/8):

$$X \sim N(74, 7^2) \Rightarrow Z = \frac{X-74}{7} \sim N(0,1)$$

$$P(X < x) = 0.88$$

$$P(Z < z) = 0.88$$



Normal Distribution (35/41)

$$P(Z < z) = 0.88$$

z	.06	.07	.08	.09
0.0	0.5239	0.5279	0.5319	0.5359
0.1	0.5636	0.5675	0.5714	0.5753
0.2	0.6026	0.6064	0.6103	0.6141
0.3	0.6406	0.6443	0.6480	0.6517
0.4	0.6772	0.6808	0.6844	0.6879
0.5	0.7123	0.7157	0.7190	0.7224
0.6	0.7454	0.7486	0.7517	0.7549
0.7	0.7764	0.7794	0.7823	0.7852
0.8	0.8051	0.8078	0.8106	0.8133
0.9	0.8315	0.8340	0.8365	0.8389
1.0	0.8554	0.8577	0.8599	0.8621
1.1	0.8770	0.8790	0.8810	0.8830
1.2	0.8962	0.8980	0.8997	0.9015

Normal Distribution (35/41)

$$z \cong 1.18$$

z	.06	.07	.08	.09
0.0	0.5239	0.5279	0.5319	0.5359
0.1	0.5636	0.5675	0.5714	0.5753
0.2	0.6026	0.6064	0.6103	0.6141
0.3	0.6406	0.6443	0.6480	0.6517
0.4	0.6772	0.6808	0.6844	0.6879
0.5	0.7123	0.7157	0.7190	0.7224
0.6	0.7454	0.7486	0.7517	0.7549
0.7	0.7764	0.7794	0.7823	0.7852
0.8	0.8051	0.8078	0.8106	0.8133
0.9	0.8315	0.8340	0.8365	0.8389
1.0	0.8554	0.8577	0.8599	0.8621
1.1	0.8770	0.8790	0.8810	0.8830
1.2	0.8962	0.8980	0.8997	0.9015

Normal Distribution (35/41)

Example5 (7/8):

$$X \sim N(74, 7^2) \Rightarrow Z = \frac{X-74}{7} \sim N(0,1)$$

$P(X < x) = 0.88$

$P(Z < z) = 0.88$

From the Table, $P(Z < 1.18)$ has the closest value to 0.88, so the desired z value is 1.18.

$\because x = \sigma z + \mu$

$\therefore x = (7)(1.18) + 74 = 82.26$

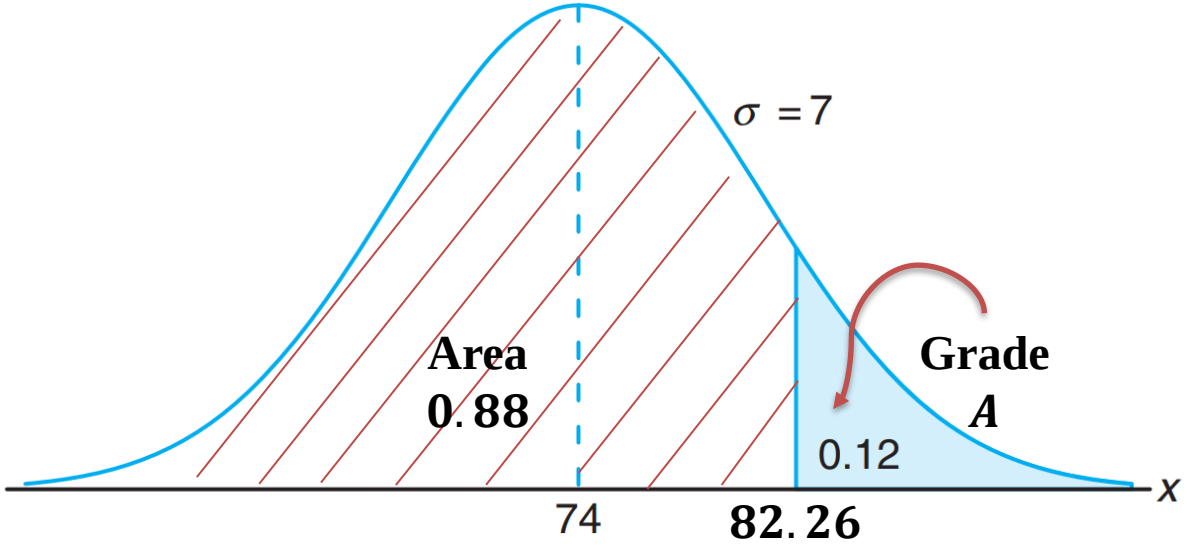
Normal Distribution (35/41)

Example5 (8/8):

$$X \sim N(74, 7^2) \Rightarrow Z = \frac{X-74}{7} \sim N(0,1)$$

$\therefore x = (7)(1.18) + 74 = 82.26.$

Therefore, the lowest A is 83 and the highest B is 82.



Normal Distribution (36/41)

Example6 (1/6):

The average life of a certain type of small motor is 10 years with a standard deviation of 2 years. The manufacturer replaces free all motors that fail while under guarantee. If he is willing to replace only 3% of the motors that fail, how long a guarantee should be offered? Assume that the lifetime of a motor follows a normal distribution.

Normal Distribution (36/41)

Example6 (2/6):

$$X \sim N(10, 2^2) \Rightarrow Z = \frac{X-10}{2} \sim N(0,1)$$

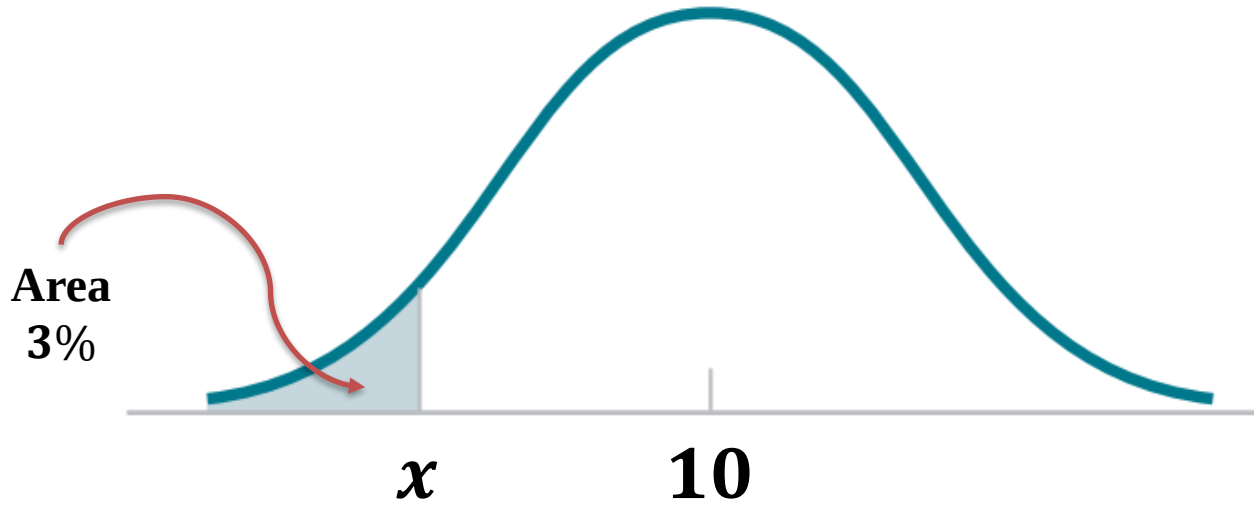
The average life of a certain type of small motor is 10 years with a **standard deviation of 2 years**. The manufacturer replaces free all motors that fail while under guarantee. If he is willing to replace only 3% of the motors that fail, how long a guarantee should be offered? Assume that the lifetime of a motor follows a normal distribution.

Normal Distribution (36/41)

Example6 (3/6):

$$X \sim N(10, 2^2) \Rightarrow Z = \frac{X-10}{2} \sim N(0,1)$$

The manufacturer replaces free all motors that fail while under guarantee. If he is willing to replace only 3% of the motors that fail, how long a guarantee should be offered? Assume that the lifetime of a motor follows a normal distribution.



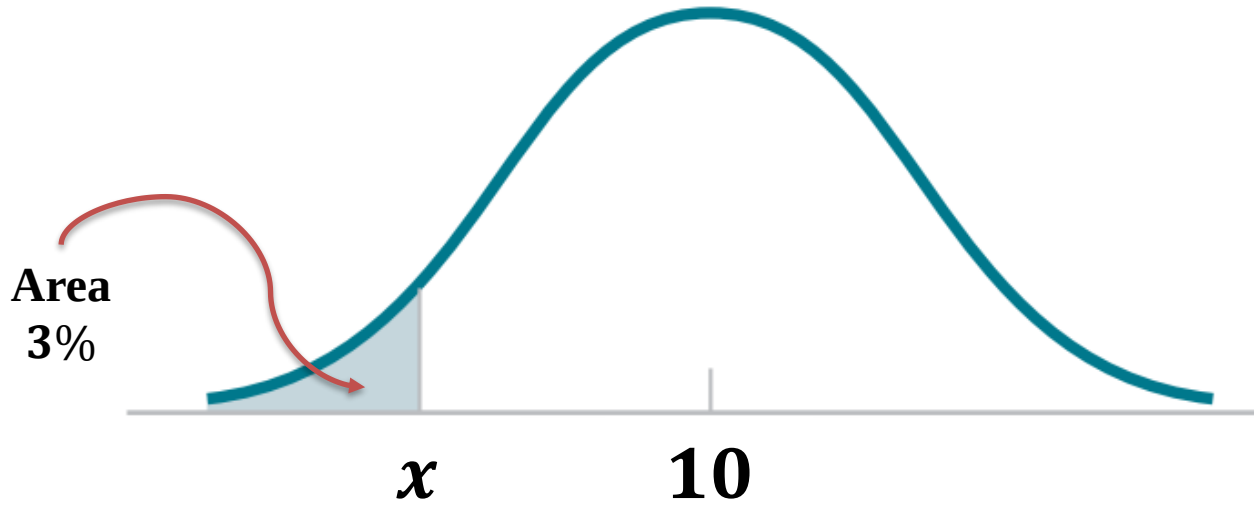
Normal Distribution (36/41)

Example6 (4/6):

$$X \sim N(10, 2^2) \Rightarrow Z = \frac{X-10}{2} \sim N(0,1)$$

$$P(X < x) = 0.03$$

$$P(Z < z) = 0.03$$



Normal Distribution (36/41)

$$P(Z < z) = 0.03$$

z	.04	.05	.06	.07	.08	.09
−2.4	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
−2.3	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
−2.2	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
−2.1	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
−2.0	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
−1.9	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
−1.8	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
−1.7	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367

Normal Distribution (36/41)

$$z \cong -1.88$$

z	.04	.05	.06	.07	.08	.09
−2.4	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
−2.3	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
−2.2	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
−2.1	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
−2.0	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
−1.9	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
−1.8	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
−1.7	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367

Normal Distribution (36/41)

Example6 (5/6):

$$X \sim N(10, 2^2) \Rightarrow Z = \frac{X-10}{2} \sim N(0,1)$$

$P(X < x) = 0.03$

$P(Z < z) = 0.03$

From the Table, $P(Z < -1.88)$ has the closest value to 0.03, so the desired z value is -1.18.

$\because x = \sigma z + \mu$

$\therefore x = (2)(-1.18) + 10 = 6.24$ years

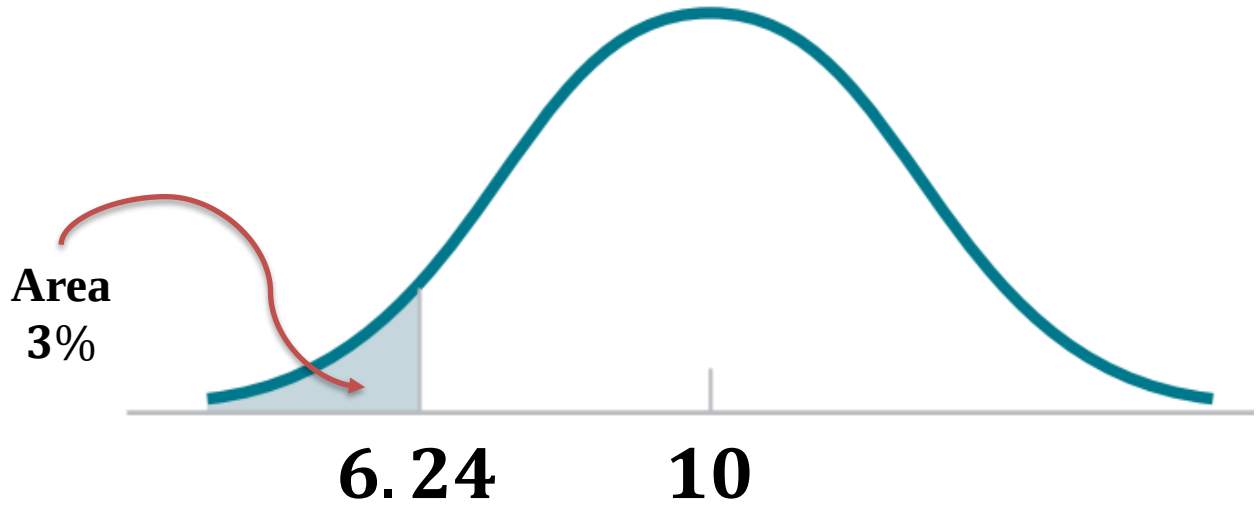
Normal Distribution (36/41)

Example6 (6/6):

$$X \sim N(10, 2^2) \Rightarrow Z = \frac{X-10}{2} \sim N(0,1)$$

$$P(X < x) = 0.03$$

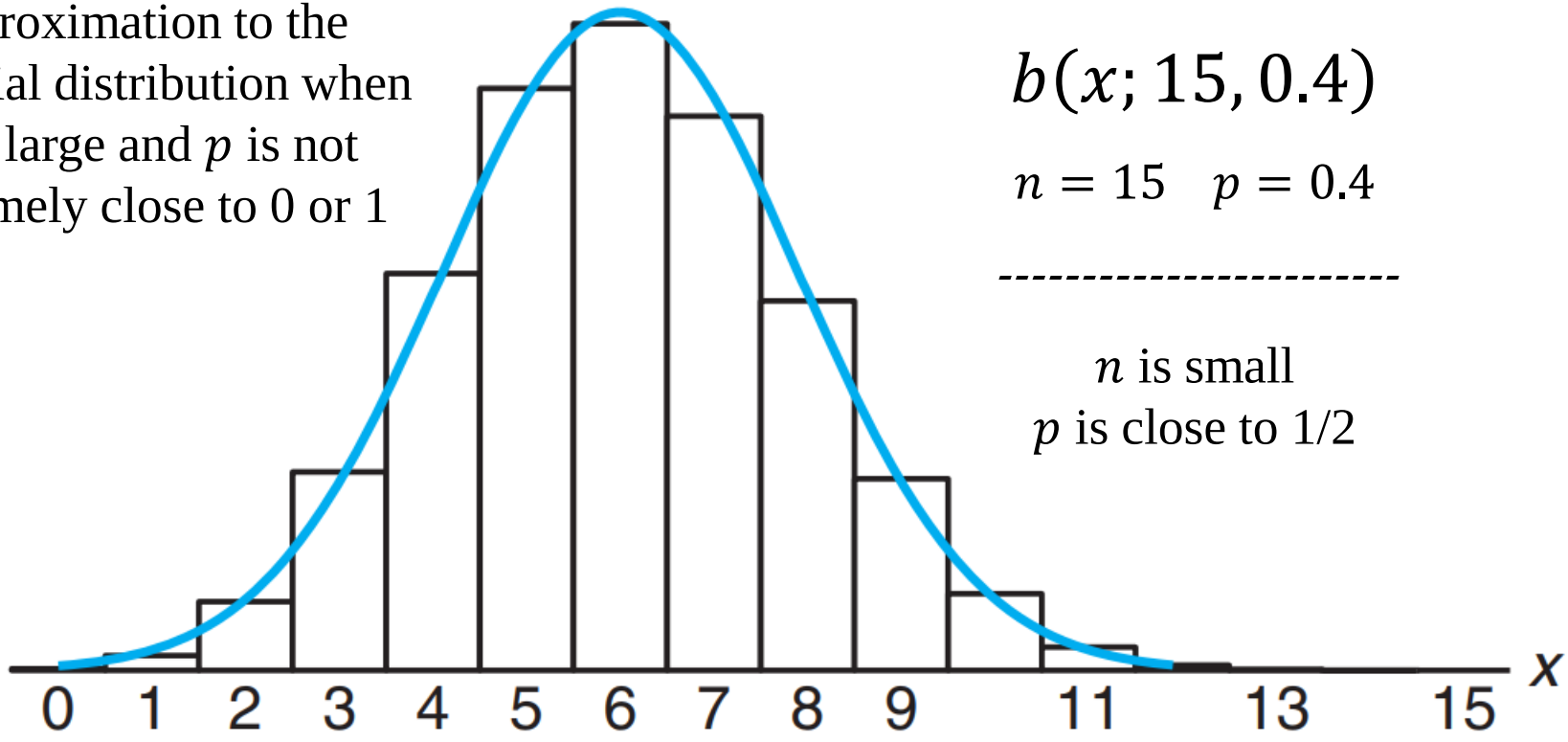
$$x = (2)(-1.18) + 10 = 6.24 \text{ years}$$



Normal Distribution (37/41)

Normal Approximation to the Binomial Distribution (1/20):

Provides a very accurate approximation to the binomial distribution when n is large and p is not extremely close to 0 or 1



$$b(x; 15, 0.4)$$

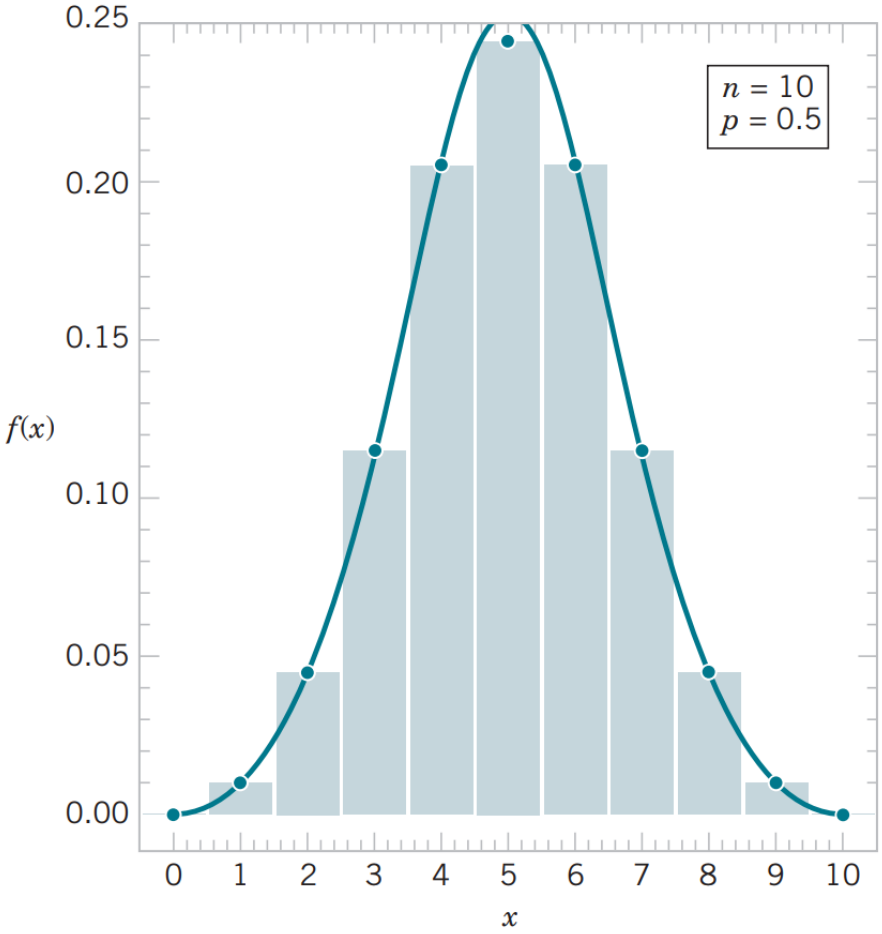
$n = 15 \quad p = 0.4$

---

n is small
p is close to 1/2

Normal Distribution (37/41)

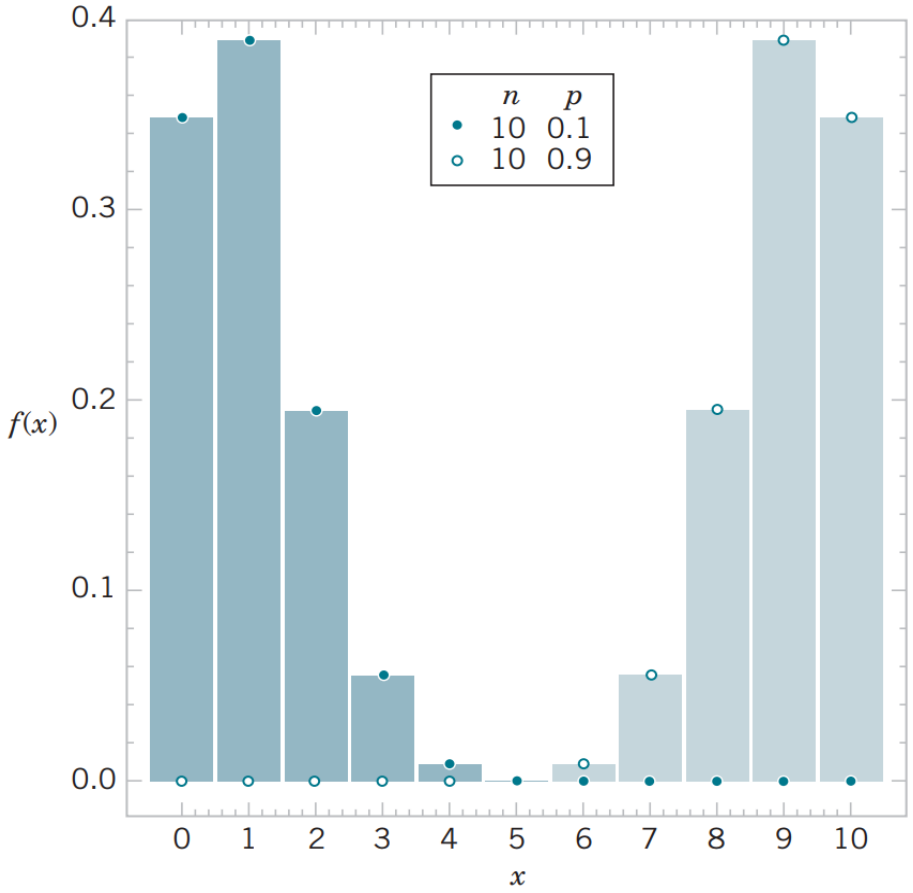
Normal Approximation to the Binomial Distribution (2/20):



Normal Distribution (37/41)

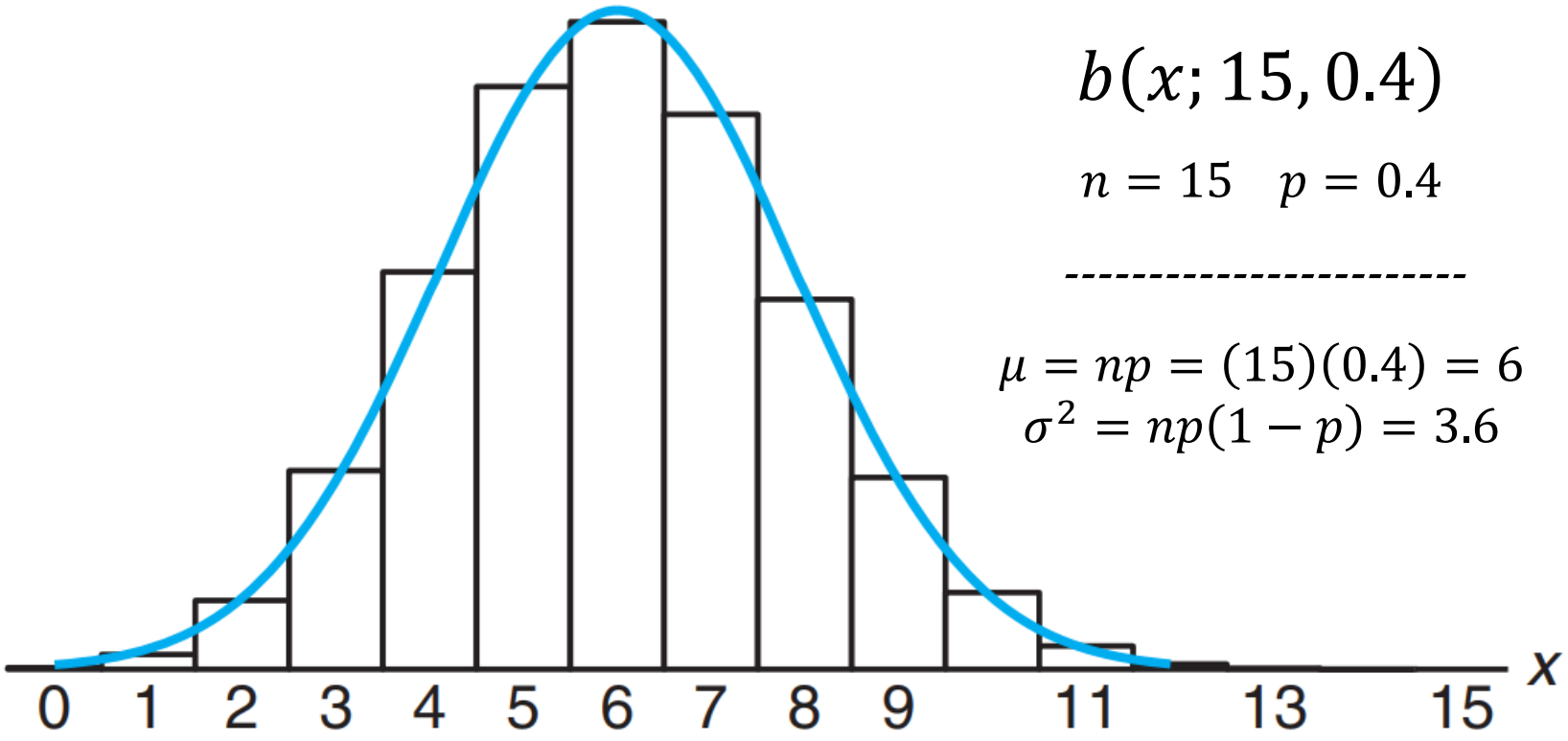
Normal Approximation to the Binomial Distribution (3/20):

Binomial distribution is not symmetrical if n is small and p is near 0 or 1



Normal Distribution (37/41)

Normal Approximation to the Binomial Distribution (4/20):



$$b(x; 15, 0.4)$$

$$n = 15 \quad p = 0.4$$

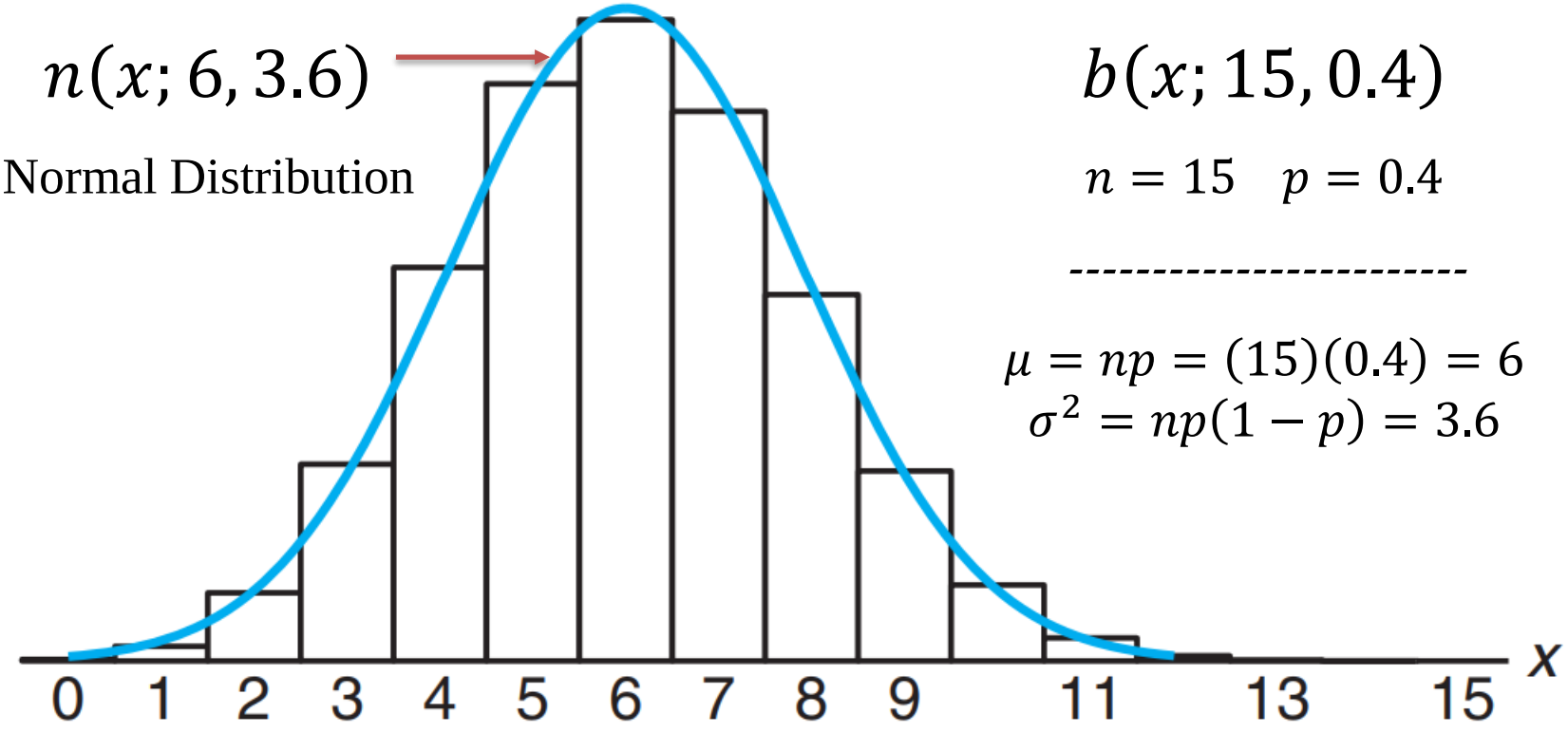
$$\mu = np = (15)(0.4) = 6$$
$$\sigma^2 = np(1-p) = 3.6$$

Normal Distribution (37/41)

Normal Approximation to the Binomial Distribution (5/20):

$n(x; 6, 3.6)$

Normal Distribution



$b(x; 15, 0.4)$

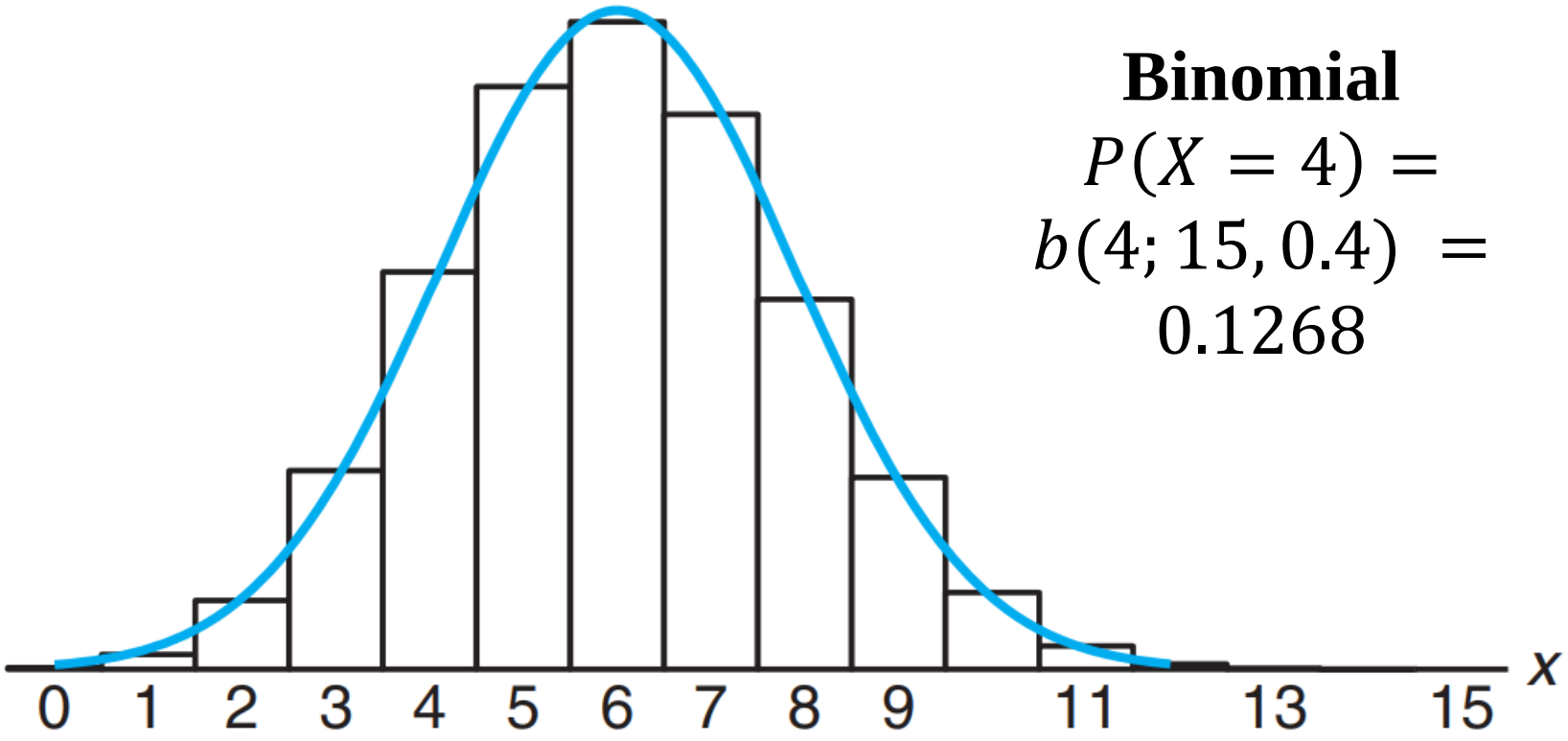
$n = 15 \quad p = 0.4$

$\mu = np = (15)(0.4) = 6$

$\sigma^2 = np(1-p) = 3.6$

Normal Distribution (37/41)

Normal Approximation to the Binomial Distribution (6/20):

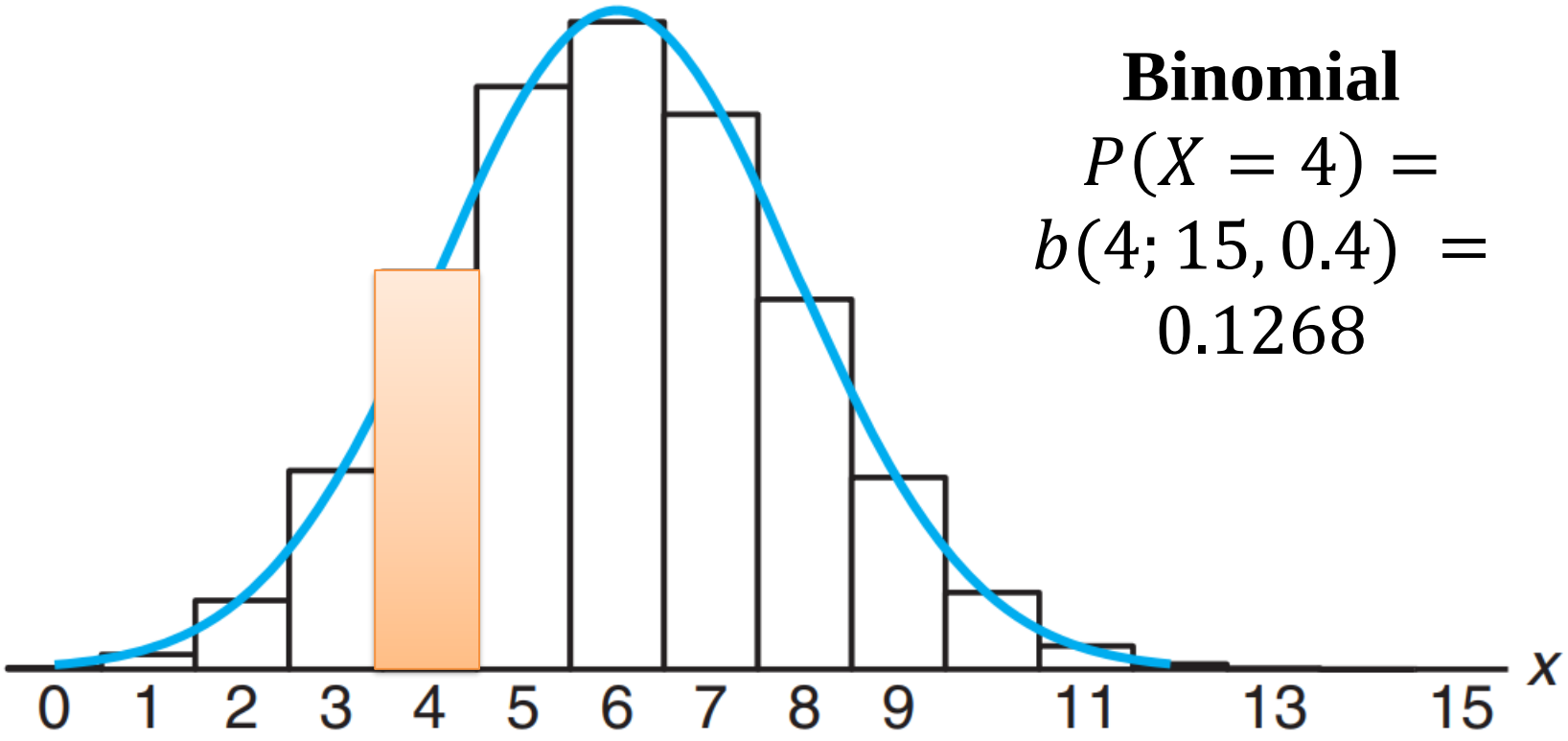


Binomial

$$P(X = 4) = b(4; 15, 0.4) = 0.1268$$

Normal Distribution (37/41)

Normal Approximation to the Binomial Distribution (7/20):

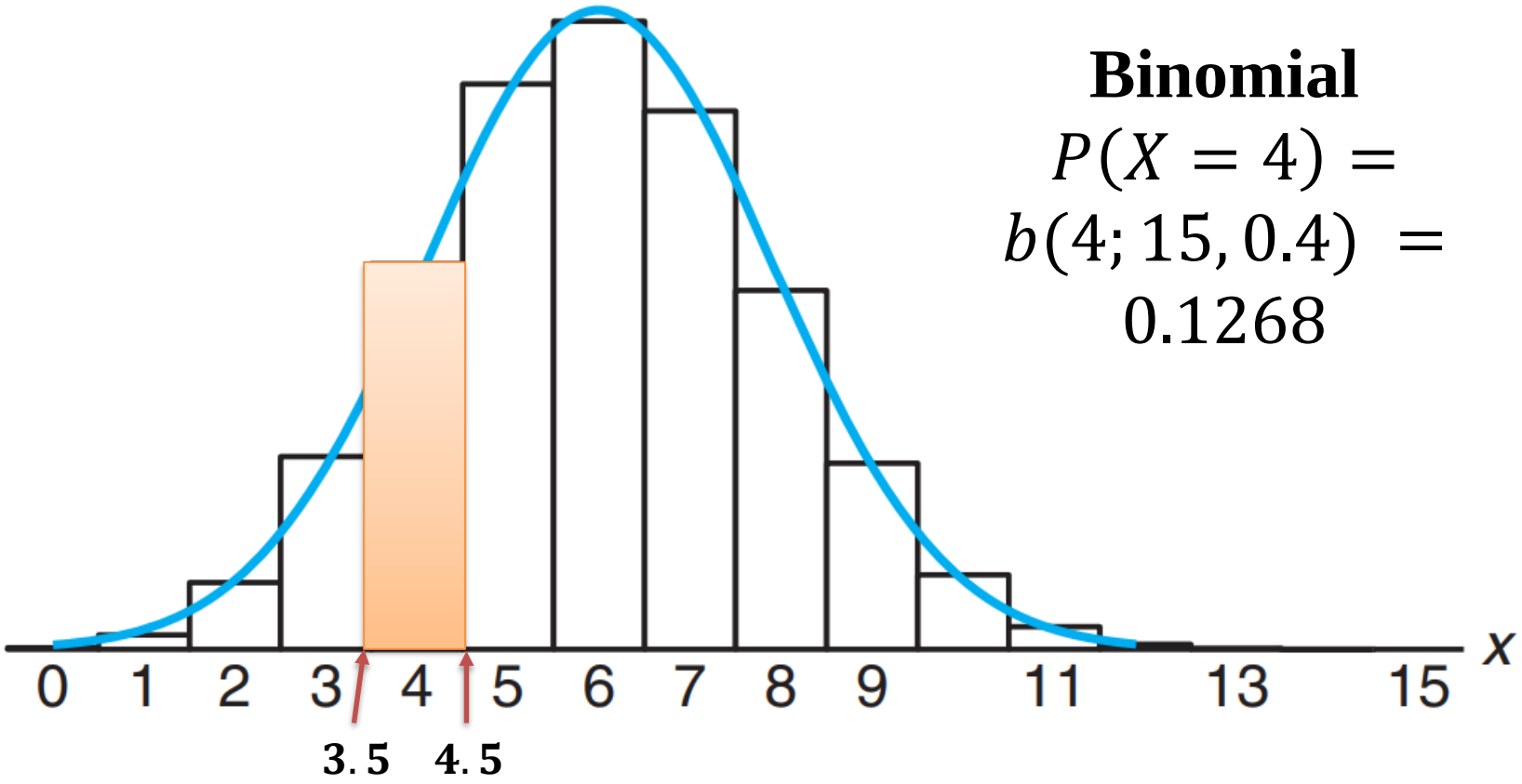


Binomial

$$P(X = 4) = b(4; 15, 0.4) = 0.1268$$

Normal Distribution (37/41)

Normal Approximation to the Binomial Distribution (8/20):

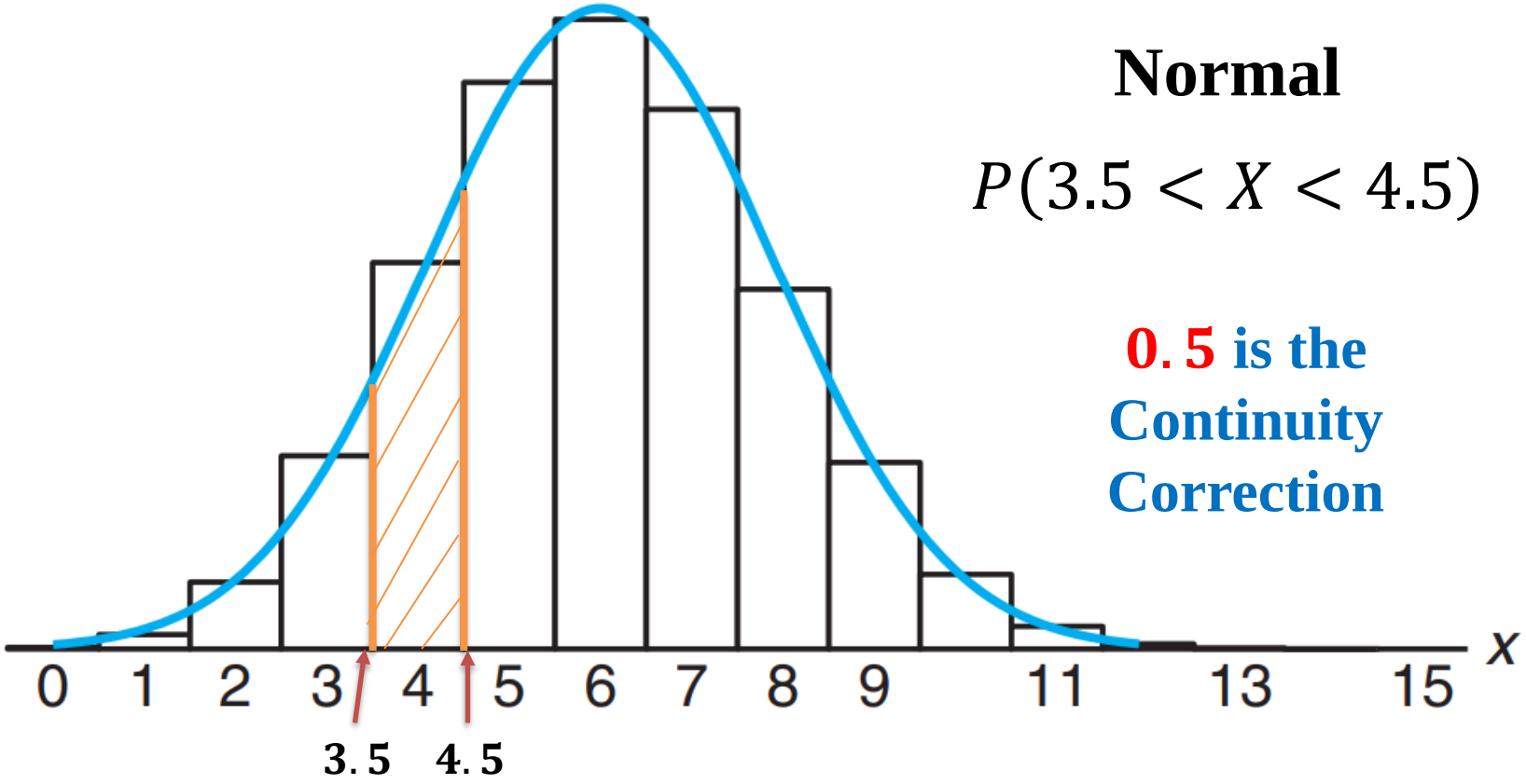


Binomial

$$P(X = 4) = b(4; 15, 0.4) = 0.1268$$

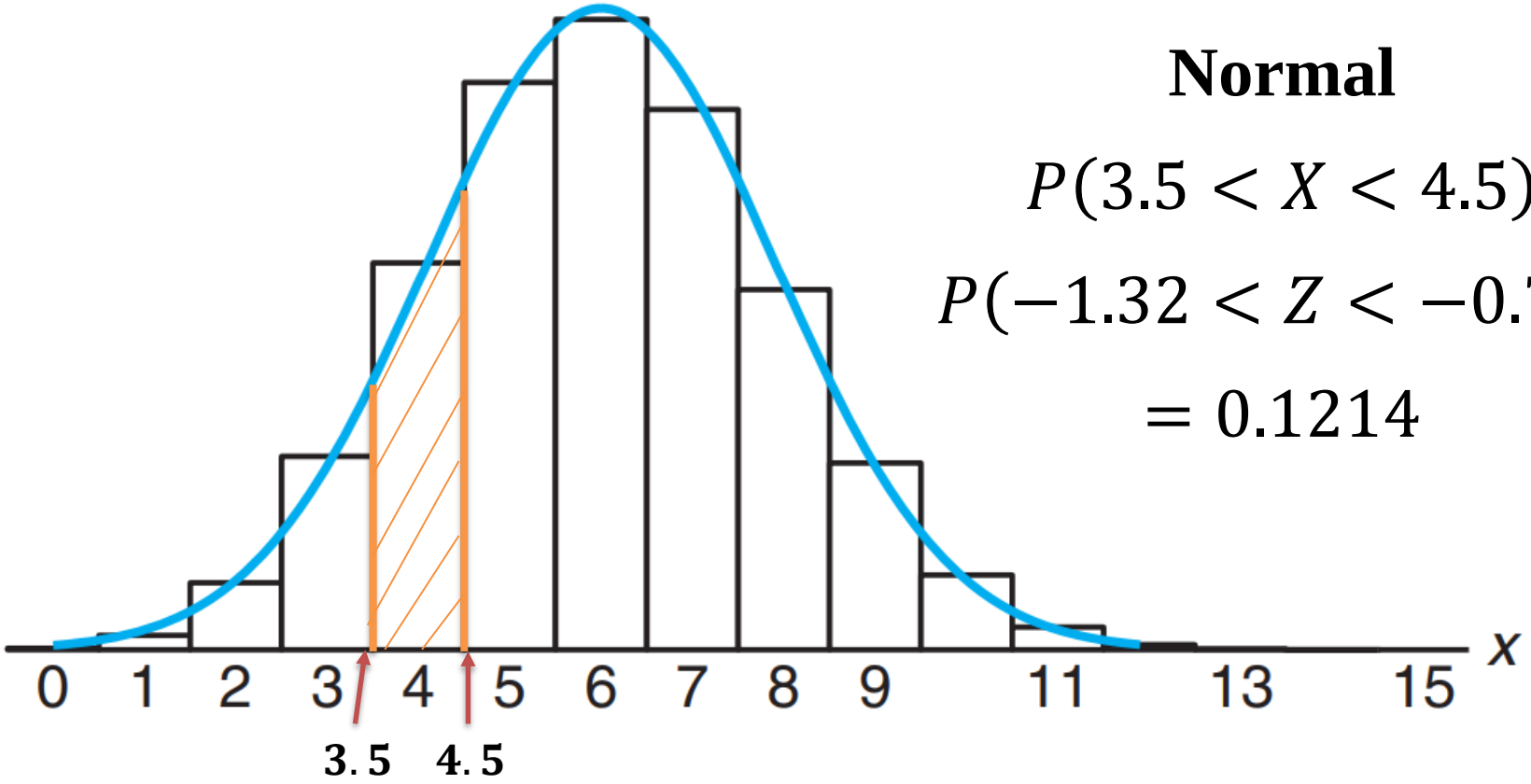
Normal Distribution (37/41)

Normal Approximation to the Binomial Distribution (9/20):



Normal Distribution (37/41)

Normal Approximation to the Binomial Distribution (10/20):



Normal

$$P(3.5 < X < 4.5)$$

$$P(-1.32 < Z < -0.79)$$

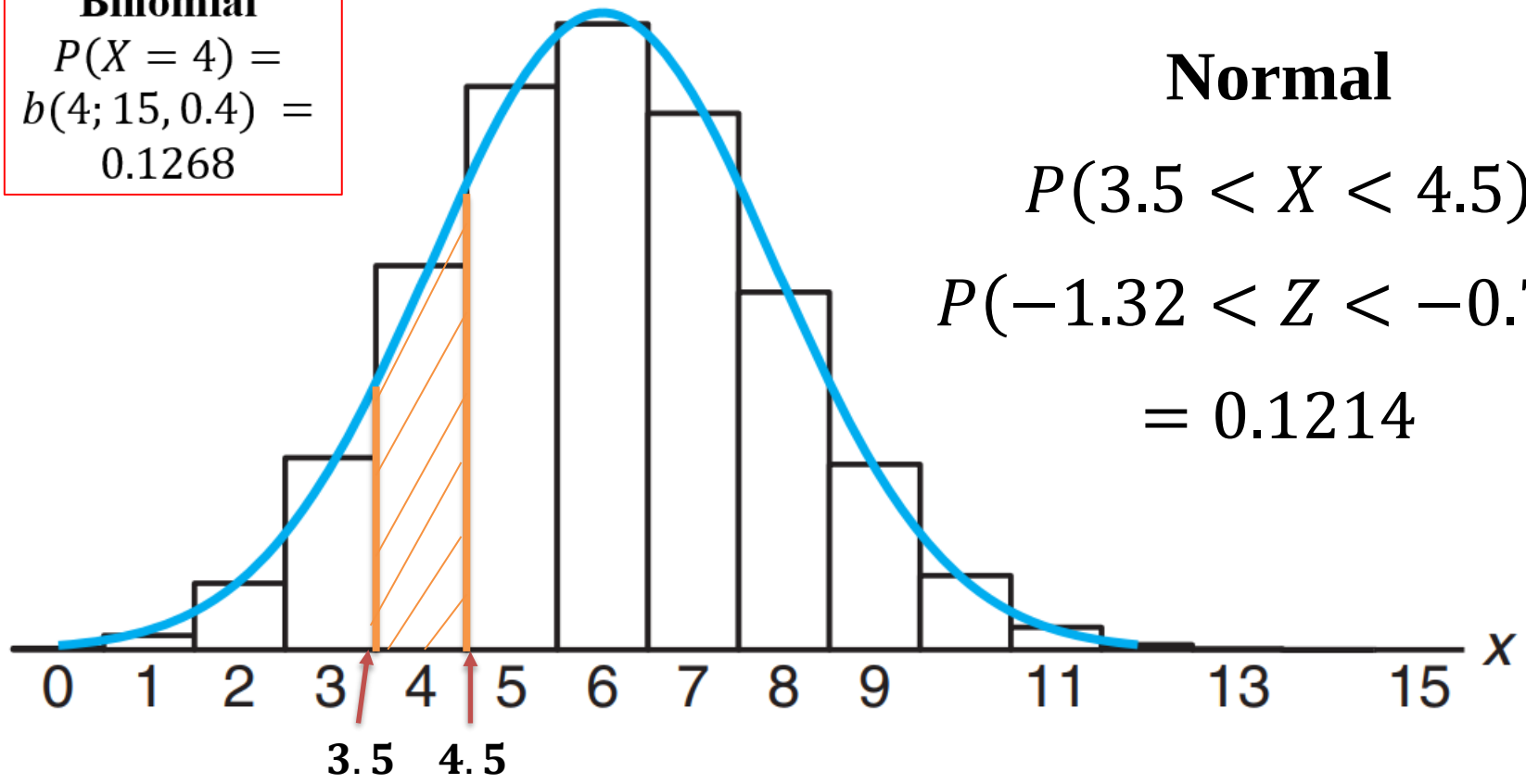
$$= 0.1214$$

Normal Distribution (37/41)

Normal Approximation to the Binomial Distribution (10/20):

Binomial

$P(X = 4) = b(4; 15, 0.4) = 0.1268$



Normal

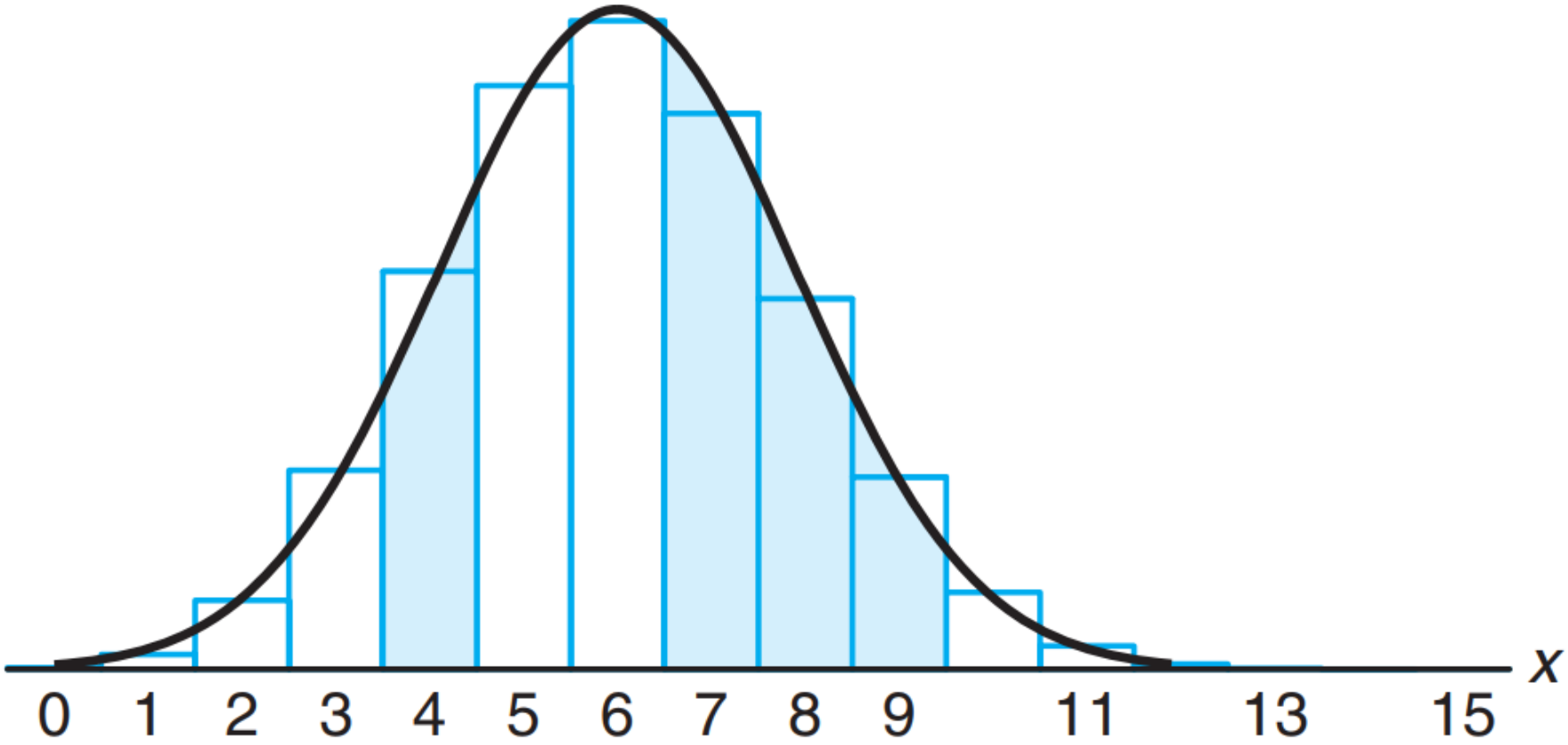
$$P(3.5 < X < 4.5)$$

$$P(-1.32 < Z < -0.79)$$

$$= 0.1214$$

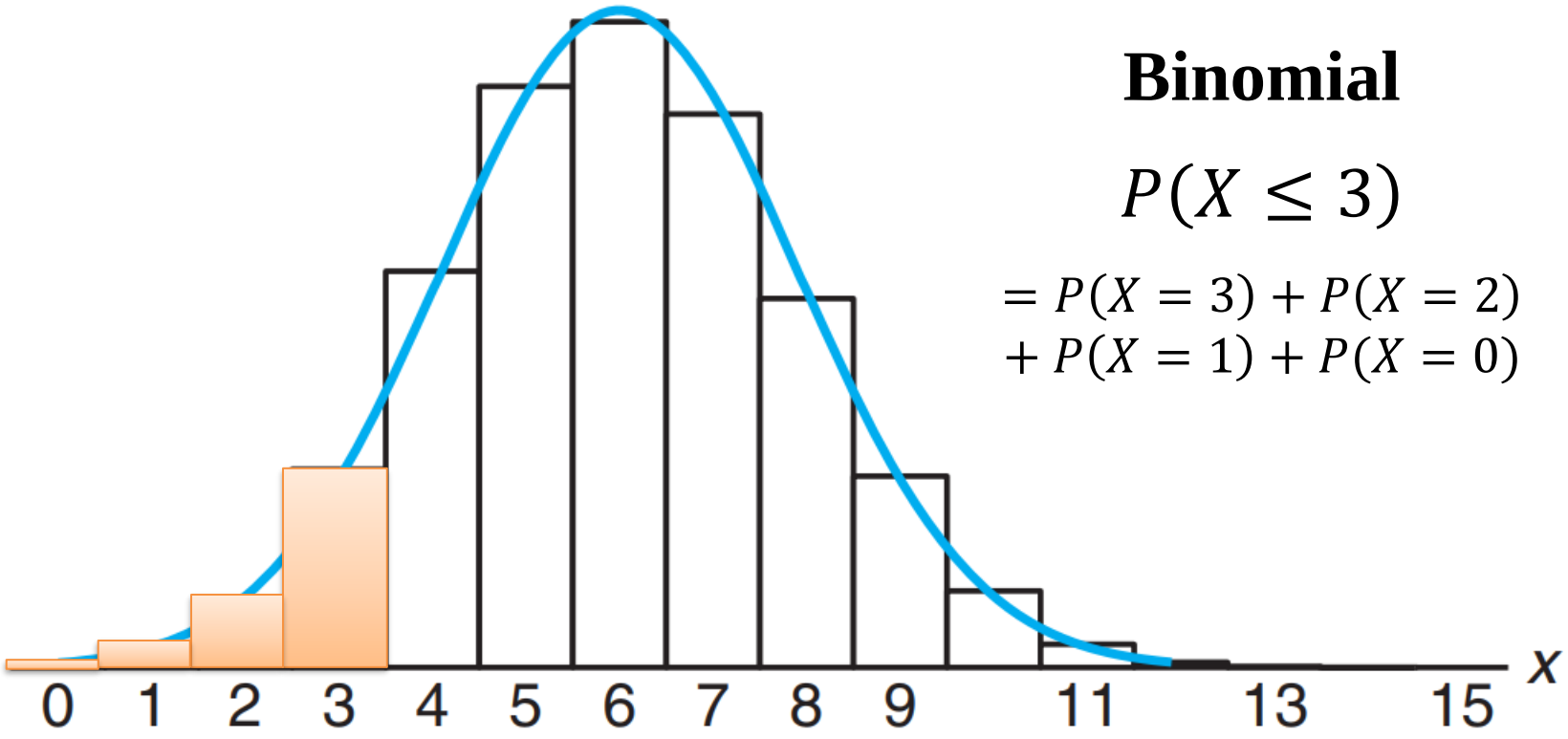
Normal Distribution (37/41)

Normal Approximation to the Binomial Distribution (11/20):



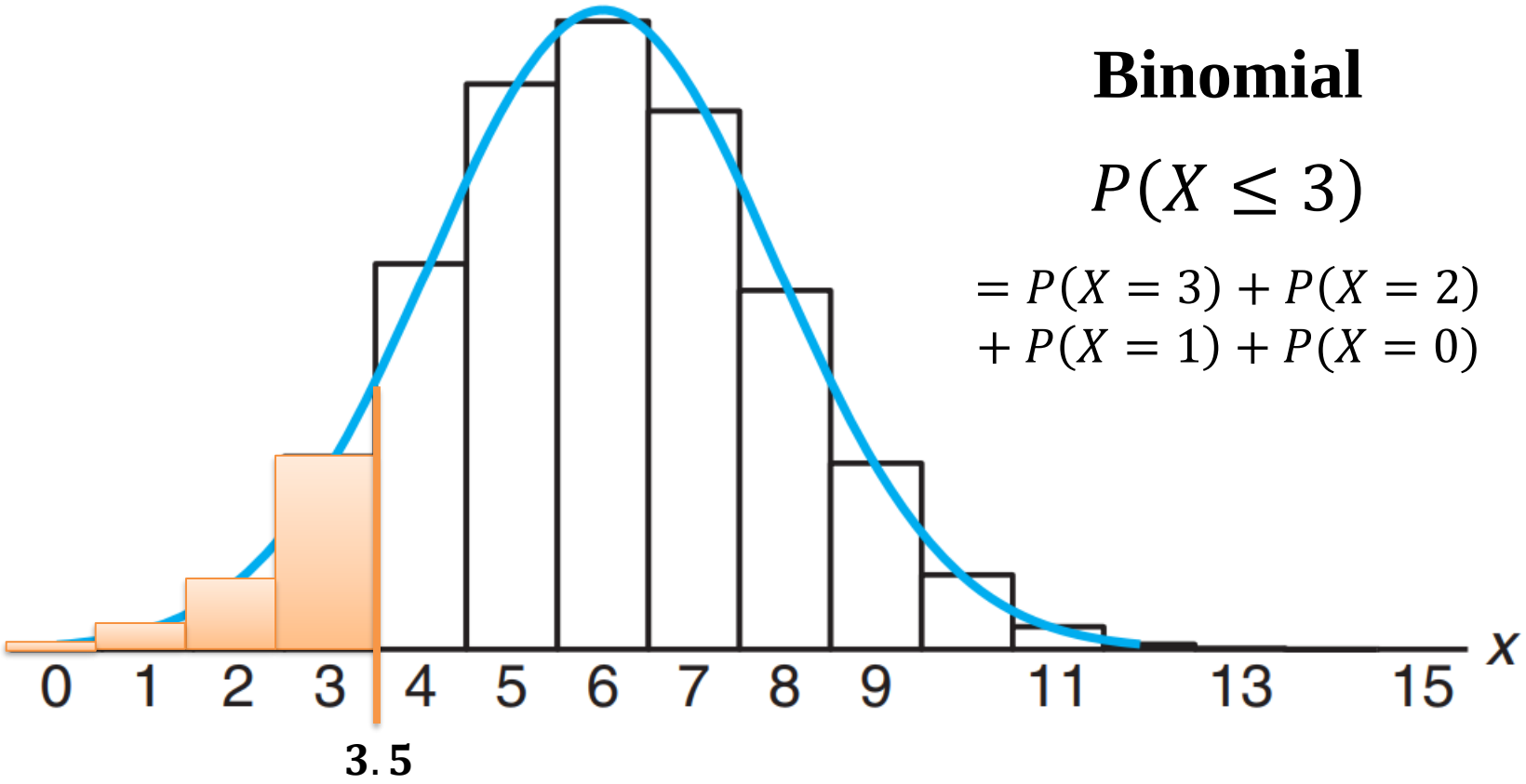
Normal Distribution (37/41)

Normal Approximation to the Binomial Distribution (12/20):



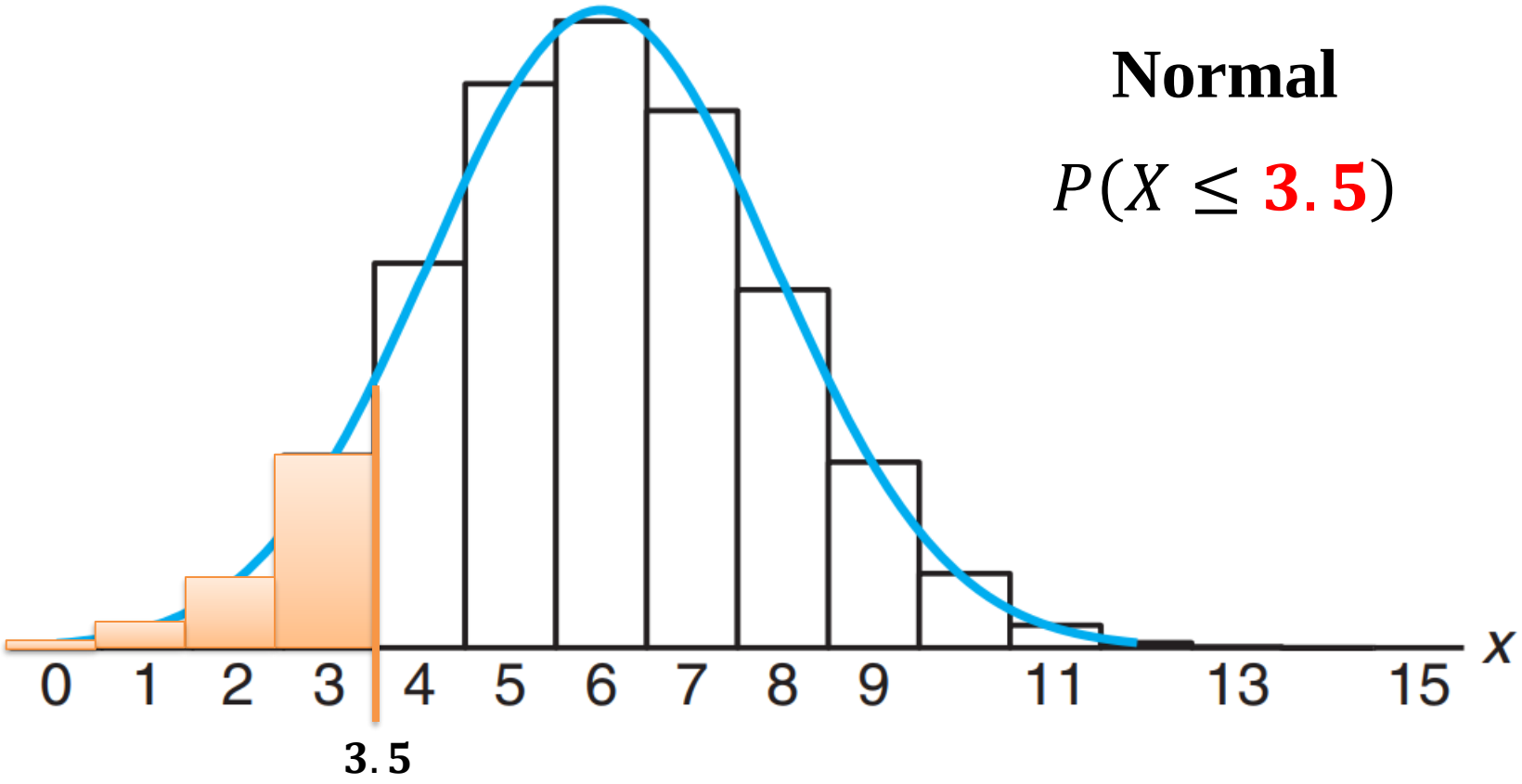
Normal Distribution (37/41)

Normal Approximation to the Binomial Distribution (13/20):



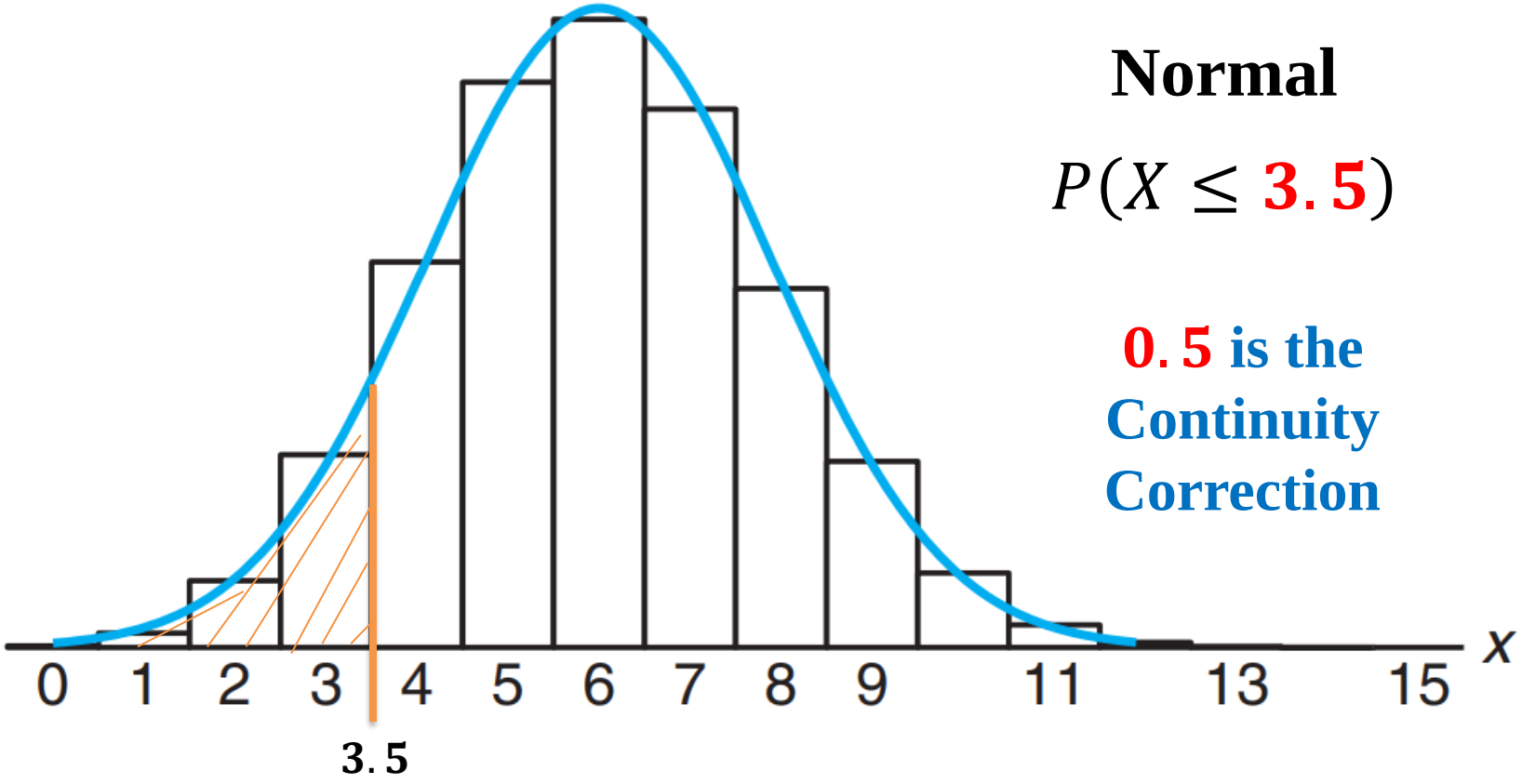
Normal Distribution (37/41)

Normal Approximation to the Binomial Distribution (14/20):



Normal Distribution (37/41)

Normal Approximation to the Binomial Distribution (15/20):



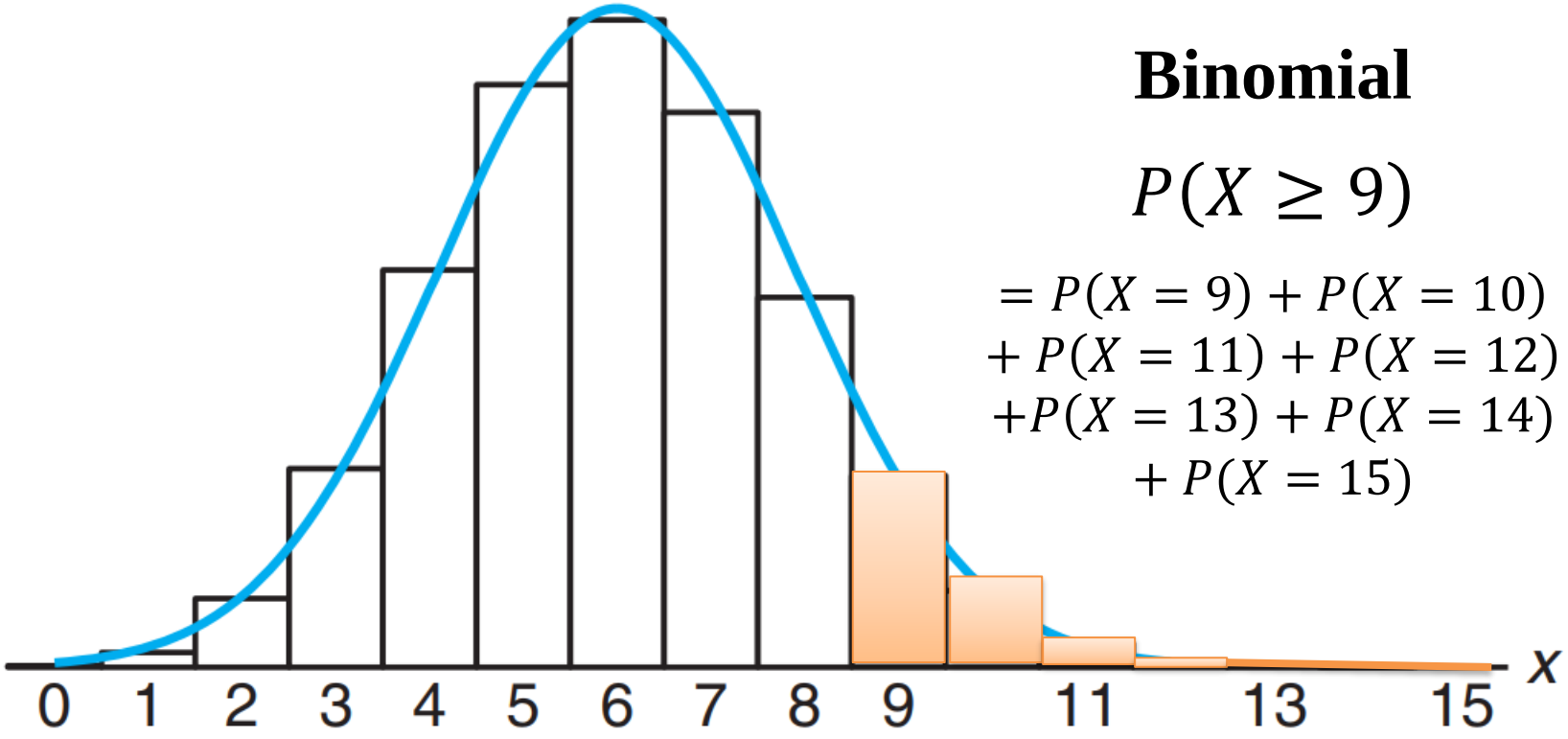
Normal

$P(X \leq 3.5)$

0.5 is the Continuity Correction

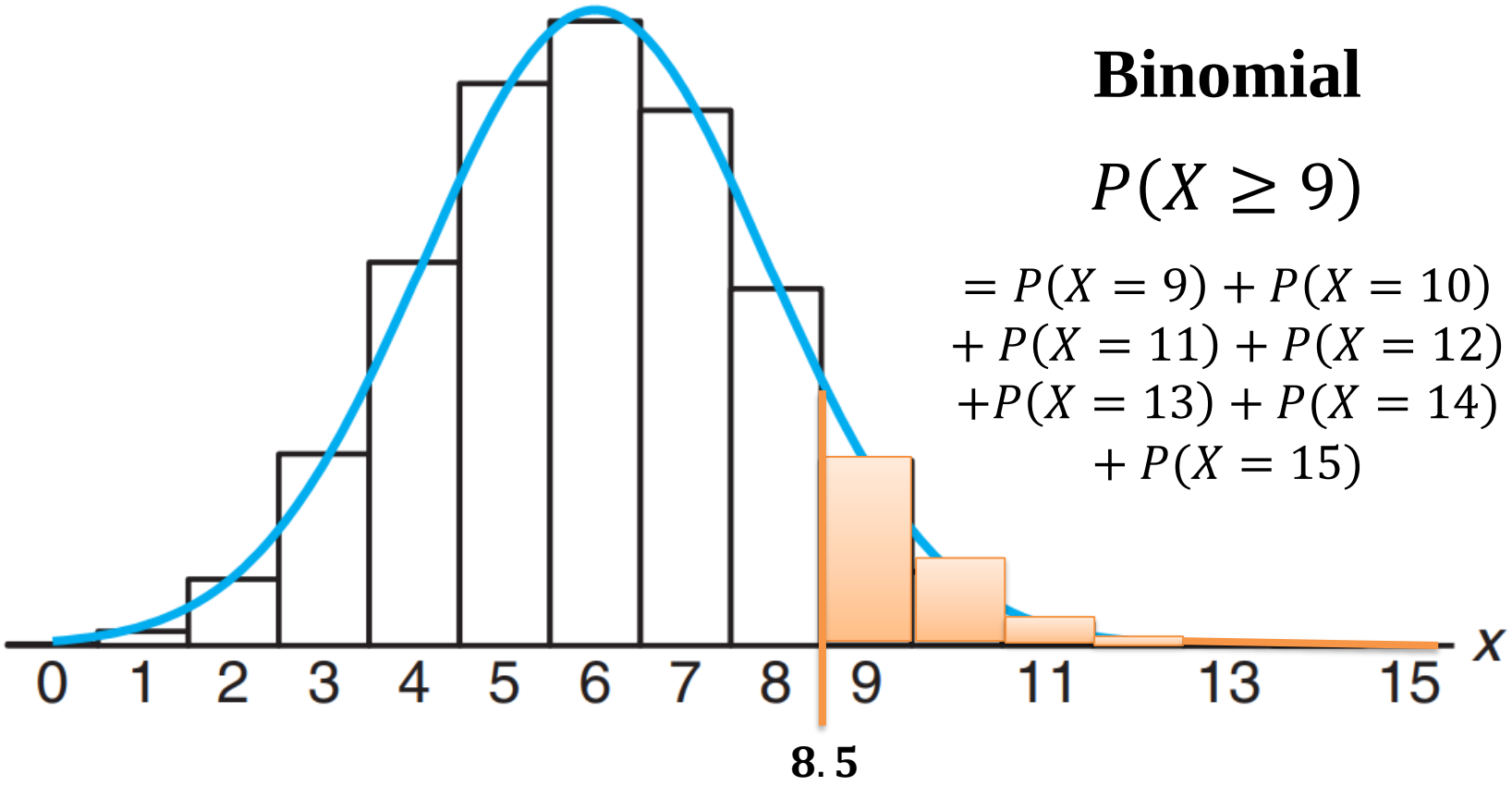
Normal Distribution (37/41)

Normal Approximation to the Binomial Distribution (16/20):



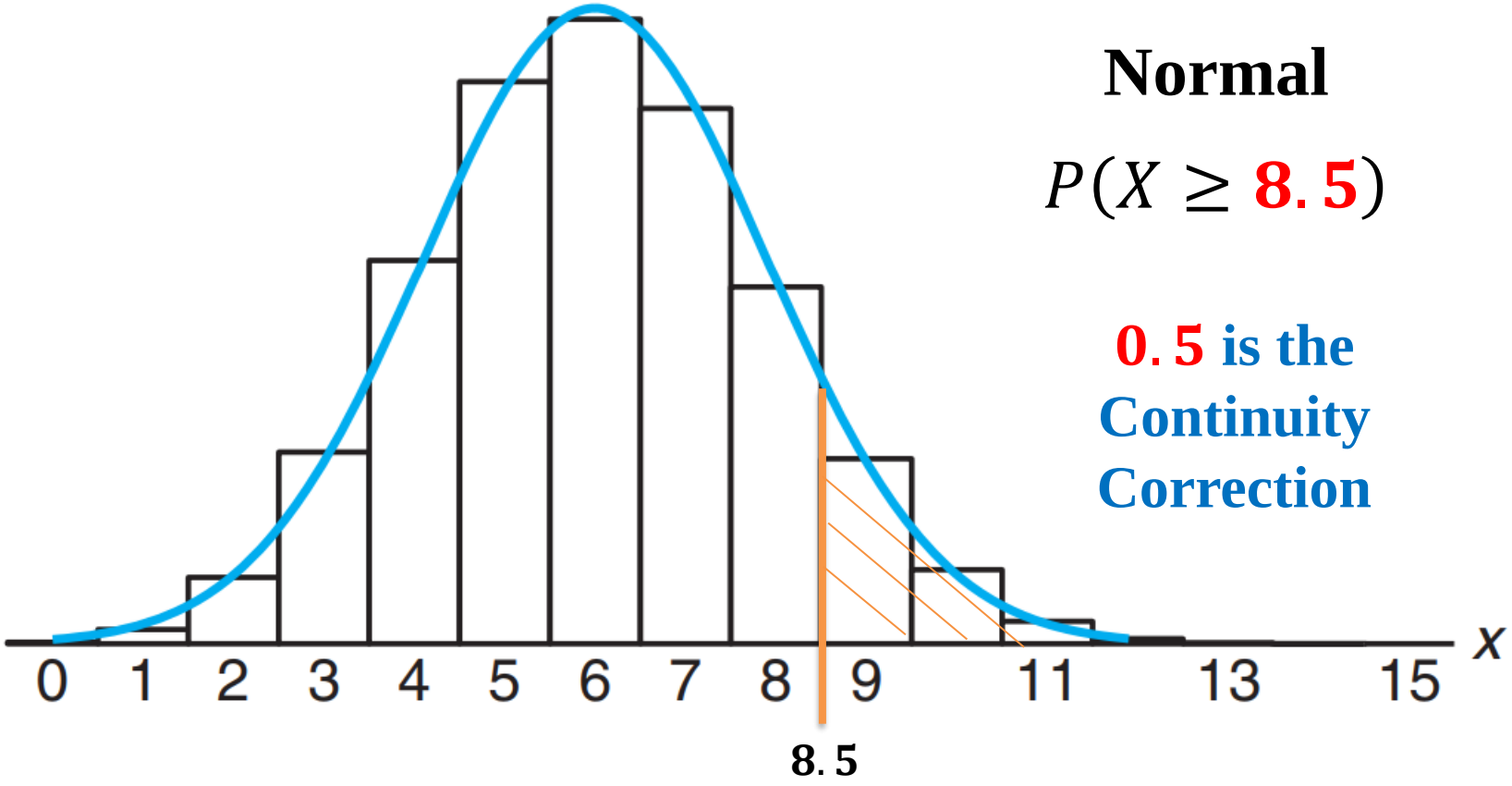
Normal Distribution (37/41)

Normal Approximation to the Binomial Distribution (17/20):



Normal Distribution (37/41)

Normal Approximation to the Binomial Distribution (18/20):



Normal Distribution (37/41)

Normal Approximation to the Binomial Distribution (19/20):

If X is a binomial random variable with parameters n and p,

Because $\mu = np$ and $\sigma^2 = np(1-p)$ for binomial, then

$$Z = \frac{X - np}{\sqrt{np(1-p)}}$$

is a standard normal random variable with mean = 0 and variance = 1.

Provides a very accurate approximation to the binomial distribution when n is large and p is not extremely close to 0 or 1

Normal Distribution (37/41)

Normal Approximation to the Binomial Distribution (20/20):

To approximate a binomial probability with a normal distribution, a *continuity correction* is applied as follows:

$$P(X \leq x) \approx P(X \leq x + 0.5) \approx P\left(Z \leq \frac{(x + 0.5) - np}{\sqrt{np(1-p)}}\right)$$

and

$$P(X \geq x) \approx P(X \geq x - 0.5) \approx P\left(Z \geq \frac{(x - 0.5) - np}{\sqrt{np(1-p)}}\right)$$

Normal Distribution (38/41)

Example1 (1/6):

A coin is tossed 400 times. Use the normal curve approximation to find the probability of obtaining

(a) between 185 and 210 heads inclusive;
(b) exactly 205 heads;
(c) fewer than 176 or more than 227 heads.

Normal Distribution (38/41)

Example1 (2/6):

$b(n = 400, p = 0.5)$

A coin is tossed 400 times. Use the normal curve approximation to find the probability of obtaining

(a) between 185 and 210 heads inclusive;
(b) exactly 205 heads;
(c) fewer than 176 or more than 227 heads.

$$\mu = np = (400)(0.5) = 200$$

$$\sigma^2 = np(1-p) = (400)(0.5)(0.5) = 100$$

$$\sigma = \sqrt{100} = 10$$

Normal Distribution (38/41)

Example1 (3/6):

$$b(n = 400, p = 0.5),\ \mu = 200,\ \sigma = 10$$

(a) between 185 and 210 heads inclusive.

$$P(185 \le X \le 210) \approx P\left(\frac{184.5 - 200}{10} \le Z \le \frac{210.5 - 200}{10}\right)$$

$$\approx P(-1.55 \le Z \le 1.05) = P(Z \le 1.05) - P(Z \le -1.55)$$

$$= 0.8531 - 0.0606 = 0.7925$$

Normal Distribution (38/41)

Example1 (4/6):

$$b(n = 400, p = 0.5),\ \mu = 200,\ \sigma = 10$$

(b) exactly 205 heads.

$$P(X = 205) \approx P\left(\frac{204.5 - 200}{10} < Z < \frac{205.5 - 200}{10}\right)$$

$$\approx P(0.45 < Z < 0.55) = P(Z < 0.55) - P(Z < 0.45)$$

$$= 0.7088 - 0.6736 = 0.0352$$

Normal Distribution (38/41)

Example1 (5/6):

$$b(n = 400, p = 0.5),\ \mu = 200,\ \sigma = 10$$

(c) fewer than 176 or more than 227 heads.

$$P(X < 176) = P(X \leq 175)$$
$$+$$
$$P(X > 227) = P(X \geq 228)$$

Binomial is Discrete

Normal Distribution (38/41)

Example1 (5/6):

$$b(n = 400, p = 0.5),\ \mu = 200,\ \sigma = 10$$

(c) fewer than 176 or more than 227 heads.

$$P(X < 176) = P(X \leq 175)$$

$$+$$

$$P(X > 227) = P(X \geq 228)$$

Binomial is Discrete

Normal Distribution (38/41)

Example1 (6/6):

$$b(n = 400, p = 0.5),\ \mu = 200,\ \sigma = 10$$

(c) fewer than 176 or more than 227 heads.

$$P(X \leq 175) + P(X \geq 228)$$

$$\approx P\left(Z \leq \frac{175.5 - 200}{10}\right) + P\left(Z \geq \frac{227.5 - 200}{10}\right)$$

$$\approx P(Z \leq -2.45) + P(Z \geq 2.75)$$

$$= P(Z \leq -2.45) + \big(1 - P(Z < 2.75)\big)$$

$$= 0.0071 + (1 - 0.9970) = 0.0101$$

Normal Distribution (39/41)

Example2 (1/5):

The probability that a patient recovers from a rare blood disease is 0.4. If 100 people are known to have contracted this disease, what is the probability that fewer than 30 survive?

Normal Distribution (39/41)

Example2 (2/5):

$$b(n = 100, p = 0.4),\ \mu = 40,\ \sigma = 4.899$$

The probability that a patient recovers from a rare blood disease is 0.4. If 100 people are known to have contracted this disease, what is the probability that fewer than 30 survive?

Let the binomial variable X represent the number of patients who survive.

$$\mu = np = (100)(0.4) = 40$$

$$\sigma^2 = np(1-p) = (100)(0.4)(0.6) = 24$$

$$\sigma = \sqrt{24} = 4.899$$

Normal Distribution (39/41)

Example2 (3/5):

$$b(n = 100, p = 0.4),\ \mu = 40,\ \sigma = 4.899$$

The probability that a patient recovers from a rare blood disease is 0.4. If 100 people are known to have contracted this disease, what is the probability that fewer than 30 survive?

$$P(X < 30) = P(X \leq 29)$$

Binomial is Discrete

Normal Distribution (39/41)

Example2 (4/5):

$$b(n = 100, p = 0.4),\ \mu = 40,\ \sigma = 4.899$$

The probability that a patient recovers from a rare blood disease is 0.4. If 100 people are known to have contracted this disease, what is the probability that fewer than 30 survive?

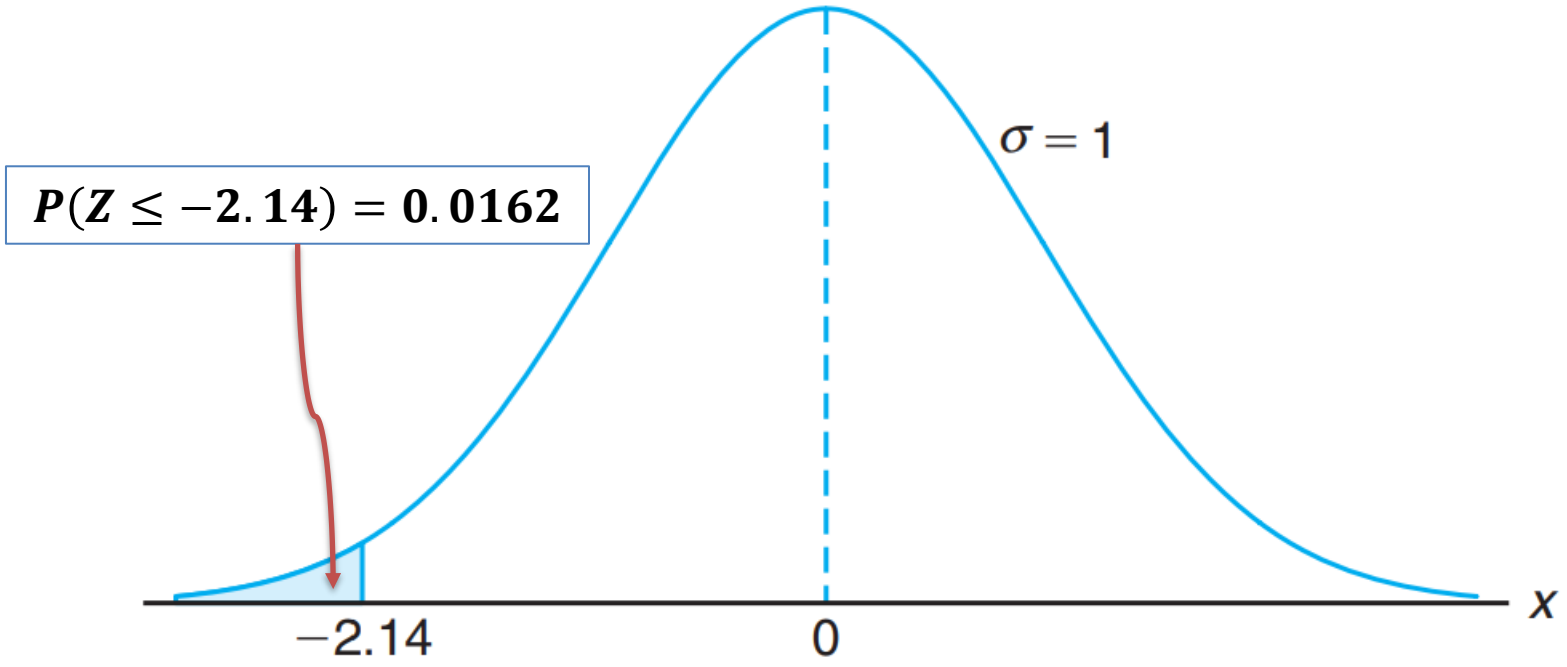
$$P(X < 30) = P(X \leq 29) \approx P\left(Z \leq \frac{29.5 - 40}{4.899}\right)$$

$$\approx P(Z \leq -2.14) = 0.0162$$

Normal Distribution (39/41)

Example2 (5/5):

$$b(n = 100, p = 0.4),\ \mu = 40,\ \sigma = 4.899$$



Normal Distribution (40/41)

Normal Approximation to the Poisson Distribution (1/2):

If X is a Poisson random variable with parameter λ,

Because $\mu = \lambda$ and $\sigma^2 = \lambda$ for Poisson, then

$$Z = \frac{X - \lambda}{\sqrt{\lambda}}$$

is a standard normal random variable

with mean = 0 and variance = 1.

Provides a very accurate approximation to the Poisson distribution when $\lambda > 5$

Normal Distribution (40/41)

Normal Approximation to the Poisson Distribution (2/2):

To approximate a Poisson probability with a normal distribution, a *continuity correction* is applied as follows:

$$P(X \leq x) = P(X \leq x + 0.5) \approx P\left(Z \leq \frac{(x + 0.5) - \lambda}{\sqrt{\lambda}}\right)$$

and

$$P(X \geq x) = P(X \geq x - 0.5) \approx P\left(Z \geq \frac{(x - 0.5) - \lambda}{\sqrt{\lambda}}\right)$$

Normal Distribution (41/41)

Example1 (1/4):

Assume that the number of asbestos particles in a squared meter of dust on a surface follows a Poisson distribution with a mean of 1000. If a squared meter of dust is analyzed, what is the probability that 950 or fewer particles are found?

Normal Distribution (41/41)

Example1 (2/4):

$\mathbf{Poisson}(\lambda = 1000)$

Assume that the number of asbestos particles in a squared meter of dust on a surface follows a **Poisson distribution with a mean of 1000**. If a squared meter of dust is analyzed, what is the probability that 950 or fewer particles are found?

Normal Distribution (41/41)

Example1 (3/4):

Poisson($\lambda = 1000$)

Assume that the number of asbestos particles in a squared meter of dust on a surface follows a Poisson distribution with a mean of 1000. If a squared meter of dust is analyzed, what is the probability that **950 or fewer** particles are found?

$$P(X \leq 950) = \sum_{x=0}^{950} \frac{e^{-1000} 1000^{x}}{x!}$$

Normal Distribution (41/41)

Example1 (4/4):

Poisson($\lambda = 1000$)

$$P(X \leq 950) = \sum_{x=0}^{950} \frac{e^{-1000}1000^{x}}{x!}$$

$$P(X \leq 950) \approx P\left(Z \leq \frac{(950 + 0.5) - 1000}{\sqrt{1000}}\right)$$

$$\approx P(Z \leq -1.57) = 0.058$$

Video Lectures

All Lectures: https://www.youtube.com/playlist?list=PLxIvc-MGOs6gW9SgkmoxE5w9vQkID1_r-

Lecture #11: https://www.youtube.com/watch?v=xtMLh4SXxZM&list=PLxIvc-MGOs6gW9SgkmoxE5w9vQkID1_r-&index=13

Thank You

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